Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

January 2018

Number: _____

Senior: _____

Junior: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space, and indicate whether you are a junior or a senior.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. (a) [5 points] Let A be a nonempty subset of $\mathbf{R}$. State the definition of what it means for $b \in \mathbf{R}$ to be an upper bound for A.

   Solution: The number $b \in \mathbb{R}$ is an upper bound for A if for every element $a \in A$ it follows that $a \leq b$.

   (b) [20 points] Suppose that A and B are nonempty subsets of the real numbers. Prove that if $\sup A < \sup B$, then there exists an element $b \in B$ that is an upper bound for A.

   Solution: Since $\sup B$ is the least upper bound for B, we know thatany number less than $\sup B$ is not an upper bound for B. Therefore, since $\sup A < \sup B$, we have that $\sup A$ is not an upper bound for B. That is, there must exists a $b \in B$ such that $b > \sup A$. Therefore for every $a \in A$, $a \leq \sup A < b$ and b is an upper bound for A.

2. (a) [5 points] State the definition of a Cauchy sequence.

   Solution: Let (a_n) be a sequence of real numbers. We say that (a_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

   (b) [5 points] State the definition of a bounded sequence.

   Solution: Let (a_n) be a sequence of real numbers. We say that (a_n) is bounded if there exists an $M > 0$ such that for evey $n \in \mathbf{N}$ it follows that $|a_n| \leq M$.

   (c) [15 points] Prove that every Cauchy sequence is bounded.

   Solution: Suppose that (a_n) is a Cauchy sequence. From the definition we know that for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$. In particular, choosing $\epsilon = 1$, we know that there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < 1$. Therefore, for all $n \geq N$ we have that $|a_n - a_N| < 1$. By the triangle inequality this implies that whenever $n \geq N$ it follows $|a_n| < |a_N| + 1$. Letting

   $$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$$

   we see that for all $n \in \mathbf{N}$ it follows that $|a_n| \leq M$ and the sequence (a_n) is bounded.

3. (a) [5 points] Let $f\colon A \to \mathbf{R}$ be a function. Using the ϵ-δ definition, define what it means for f to be continuous at $c \in A$.

   Solution: The function $f\colon A \to \mathbf{R}$ is continuous at $c \in A$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$ it follows that $|f(x) - f(c)| < \epsilon$.

   (b) [20 points] Suppose that $f\colon A \to \mathbf{R}$ is continuous at $c \in A$. Prove using the above definition that $|f|$ is continuous at c.

Solution: Let $\epsilon > 0$ be given. Since f is continuous at $c \in A$ there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$ it follows that $|f(x) - f(c)| < \epsilon$. This together with a corollary to the triangle inequality shows that whenever $|x - c| < \delta$ and $x \in A$ it follows that $||f(x)| - |f(x)|| \le |f(x) - f(c)| < \epsilon$. Therefore $|f|$ is continous are $c \in A$.

4. (a) [5 points] State the Weierstrass M-test.

Solution: For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$, and let $M_n > 0$ be a real number such that

$$|f_n(x)| \le M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

(b) [20 points] Use part (a) to show that for any $r \in (0, 1)$ the function $f(x) = \sum_{n=1}^{\infty} x^n$ is well-defined and continuous on $[-r, r]$.

Solution: Let $f_n(x) = x^n$ and $r \in (0, 1)$. Then for each $n \in \mathbf{N}$, f_n is a function defined on $[-r, r]$ and if we let $M_n = r^n > 0$ then

$$|f_n(x)| \le r^n$$

for all $x \in [-r, r]$. Further, the series

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$$

is a convergent geometric series. Thus by The Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function, call it f, on $[-r, r]$. Since each f_n is continuous on $[-r, r]$ (and in fact on all of $\mathbf{R}$) and the convergence is uniform we know that the limit function f is also continuous on $[-r, r]$.