Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 26, 2018

1. [25 points] Find an equation for the plane that passes through the point (1,3,5) and contains the line
$$x = 5t,\ y = 1 + t,\ z = 3 - t.$$

Solution: Letting $t = 0$ and $t = 1$, we find that the points $(0, 1, 3)$ and $(5, 2, 2)$ are on the plane. We also have that $(1, 3, 5)$ is on the plane. We compute the vectors

$$\langle 5,2,2\rangle - \langle 0,1,3\rangle = \langle 5,1,-1\rangle$$
$$\langle 5,2,2\rangle - \langle 1,3,5\rangle = \langle 4,-1,-3\rangle .$$

The normal vector is

$$\boldsymbol{n} = \langle 5,1,-1\rangle \times \langle 4,-1,-3\rangle = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 5 & 1 & -1 \\ 4 & -1 & -3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 5 & -1 \\ 4 & -3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix}$$
$$= -4\boldsymbol{i} + 11\boldsymbol{j} - 9\boldsymbol{k}.$$

Thus, the plane has equation $\langle -4, 11, -9\rangle \cdot \langle x-1, y-3, z-5\rangle = 0$ which is equivalent to $-4(x-1) + 11(y-3) - 9(z-5) = 0$.

2. [25 points] Let $f(x, y) = x^4 - 4xy + 2y^2$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

Solution: We compute

$$f_x(x,y) = 4x^3 - 4y, \quad f_y(x,y) = -4x + 4y.$$

Setting the first of these equations equal to 0 we have that $y = x^3$. We substitute this into the second equation and set it equal to 0 to obtain

$$-4x + 4x^3 = 0 \Leftrightarrow 4x(x^2-1) = 0 \Leftrightarrow 4x(x-1)(x+1) = 0 \Rightarrow x \in \{0, \pm 1\}.$$

With these three x values, using that $y = x^3$, we have the three critical points

$$(0,0), (1,1), (-1,-1).$$

We now compute

$$f_{xx}(x,y) = 12x^2, \quad f_{yy}(x,y) = 4, \quad f_{xy}(x,y) = f_{yx}(x,y) = -4,$$

and

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 4 \end{vmatrix} = 48x^2 - 16.$$

Then

$$D(0,0) = -16 < 0 \Rightarrow (0,0) \text{ is a saddle point,}$$
$$D(1,1) = 32 > 0 \Rightarrow (1,1) \text{ is a local min,}$$
$$D(-1,-1) = 32 > 0 \Rightarrow (-1,-1) \text{ is a local min.}$$

3. [25 points] Calculate the volume of the region in the first octant that lies both inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$.
Note. The first octant is the region where x, y and z are all ≥ 0.

Solution: The sphere and cone intersect when

$$2 = x^2 + y^2 + z^2 = x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 2x^2 + 2y^2,$$

that is, when

$$x^2 + y^2 = 1.$$

We seek the volume of the region above the cone and inside the sphere in the first octant, so we integrate $\sqrt{2 - x^2 - y^2} - \sqrt{x^2 + y^2}$. We will give a solution using polar coordinates: let $x = r\cos\theta, y = r\sin\theta$, so that $x^2 + y^2 = r^2$ and $dA = rdrd\theta$. Thus, the volume of the region is given by

$$\begin{aligned}
V &= \iint_{x^2+y^2\leq 1} \left(\sqrt{2 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dA \\
&= \int_{\theta=0}^{\frac{\pi}{2}} \int_0^1 \left(\sqrt{2 - r^2} - r\right) rdrd\theta \\
&= \left(\int_{\theta=0}^{\frac{\pi}{2}} d\theta\right) \cdot \left(\int_0^1 \left(\sqrt{2 - r^2} - r\right) rdr\right) \\
&= \frac{\pi}{2} \int_0^1 \left(\sqrt{2 - r^2} - r\right) rdr \\
&= \frac{\pi}{2} \int_0^1 \sqrt{2 - r^2}\ rdr - \frac{\pi}{2} \int_0^1 r^2 dr
\end{aligned}$$

To evaluate the first integral above, we let $u = 2 - r^2$ so that $du = -2rdr$. The second integral is evaluated directly. Thus, we have that

$$V = \frac{\pi}{2} \int_2^1 \sqrt{u} \cdot \frac{du}{-2} - \frac{\pi}{6} = -\frac{\pi}{4} \cdot \frac{2}{3} \left(1 - 2^{\frac{3}{2}}\right) - \frac{\pi}{6} = \frac{\pi}{3} \left(\sqrt{2} - 1\right).$$

4. [25 points] Compute $\int_C y\,dx + x\,dy$ where C is the boundary curve of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 16$, traversed in the counterclockwise direction.
Note. This integral may also be written as $\int_C \langle y, x\rangle \cdot d\boldsymbol{r}$

Solution: The functions $f(x, y) = y$ and $g(x, y) = x$ are defined everywhere, and have continuous partial derivatives, so by Green's Theorem, with D equal to the domain bounded by C, this integral equals

$$\iint_D \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right) dA = \iint_D (1 - 1) dA = \iint_D 0\ dA = 0.$$

5. (a) [10 points] Suppose V is a vector space. Explain what it means to say that a subset U of V is a subspace.

   Solution: It means that U is itself a vector space when given the same operations of addition and scalar multiplication as V.

   More explicitly, U should be non-empty, closed under addition and closed under scalar multiplication.

   (b) [5 points] Let $V = \mathbb{R}^2$ and let U be the subset consisting of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + 2y = 1$. Say whether U is a subspace of V or not, and justify your answer.

   Solution: A subspace must contain the zero object in the vector space, but $0+2\cdot 0 \neq 1$, so U does not contain the zero vector in $\mathbb{R}^2$. So U is *not* a subspace.

   (c) [10 points] Give two other examples of subsets of $\mathbb{R}^2$, one that *is* a subspace, and one that is *not* a subspace. Justify your answers.

   Solution: Many options.

6. (a) [5 points] Explain what it means to say that a subset S of a vector space V is a *basis* of V.

   Solution: It means that S is linear independent, and is a spanning set for V.

   (b) [20 points] Give a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

   You should explain how you know that your answer really is a basis; namely, you should relate your answer to the definition you gave in part (a) for full credit.

   Solution: The first and third vectors here form a basis for that subspace. To see that these span the subspace, we note that the second vector is a scalar multiple of the first, and the fourth vector is a linear combination of the first and third.

   To see that they are linearly independent, suppose that

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

   Then

$$\begin{bmatrix} a_1 \\ a_2 \\ 2a_1 + 2a_2 \\ -a_1 - a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and so $a_1 = a_2 = 0$. So these two vectors are linearly independent.

Alternatively, on could show that the matrix made

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 2 & -4 & 2 & 2 \\ -1 & 2 & -1 & -1 \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the column space of this matrix is spanned by the first and the third vectors above and those two vectors are linearly independent.

7. [25 points] Let A be the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Solution: First we find eigenvalues:

$$\det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ -6 & -1 & -3-\lambda \end{bmatrix} = (2-\lambda)[(2-\lambda)(-3-\lambda) - (1)(-6)]$$
$$= (2-\lambda)(-6+\lambda+\lambda^2+6)$$
$$= (2-\lambda)\lambda(\lambda+1).$$

Therefore, the eigenvalues are $\lambda = -1, 0, 2$. *[10 points for finding the eigenvalues]*

Next we find an eigenvector for each eigenvalue by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

- for $\lambda = -1$:

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 0 \\ -6 & -1 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

has solution, for example, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

- for $\lambda = 0$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

has solution, for example, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

- for $\lambda = 2$:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ -6 & -1 & -5 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

has solution, for example, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$

Therefore, examples of the required matrices are

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ -3 & -2 & -3 \end{bmatrix}.$$

8. [25 points] Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove that 0 is an eigenvalue of T if and only if the image (i.e., range) of T is *not* equal to V.

Solution: By definition, 0 is an eigenvalue of T if and only if there is a nonzero vector $\mathbf{v}$ such that $T(\mathbf{v}) = 0\mathbf{v} = \mathbf{0}$. This is the case if and only if the kernel of T contains a nonzero vector. This in turn is the case if and only if the nullity of T is more than 0. By the Rank-Nullity Theorem, we know that the rank of T plus the nullity of T is equal to the $\dim V$. Therefore, the nullity is more than 0 if and only if the rank is less than $\dim V$, which is the case if and only if the image of T is not equal to V.