Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination
◁ Multivariable Calculus and Linear Algebra ▷
January 2018

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. [25 points] Find an equation for the plane that passes through the point (1,3,5) and contains the line
$$x = 5t,\ y = 1 + t,\ z = 3 - t.$$

2. [25 points] Let $f(x, y) = x^4 - 4xy + 2y^2$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

3. [25 points] Calculate the volume of the region in the first octant that lies both inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$.
Note. The first octant is the region where x, y and z are all ≥ 0.

4. [25 points] Compute $\int_C y\,dx + x\,dy$ where C is the boundary curve of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 16$, traversed in the counterclockwise direction.
Note. This integral may also be written as $\int_C \langle y, x\rangle \cdot d\boldsymbol{r}$

5. (a) [10 points] Suppose V is a vector space. Explain what it means to say that a subset U of V is a subspace.
   (b) [5 points] Let $V = \mathbb{R}^2$ and let U be the subset consisting of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + 2y = 1$. Say whether U is a subspace of V or not, and justify your answer.
   (c) [10 points] Give two other examples of subsets of $\mathbb{R}^2$, one that *is* a subspace, and one that is *not* a subspace. Justify your answers.

6. (a) [5 points] Explain what it means to say that a subset S of a vector space V is a *basis* of V.
   (b) [20 points] Give a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors
$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix},\quad \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix},\quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix},\quad \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$
   You should explain how you know that your answer really is a basis; namely, you should relate your answer to the definition you gave in part (a) for full credit.

7. [25 points] Let A be the matrix
$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ -6 & -1 & -3 \end{bmatrix}.$$
Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

8. [25 points] Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Prove that 0 is an eigenvalue of T if and only if the image (i.e., range) of T is *not* equal to V.