Amherst College Department of Mathematics and Statistics

Solutions to Comprehensive Examination: Algebra

February, 2019

1. **(25 points)** Let G be a group, let $H \subseteq G$ be a subgroup, and define the **normalizer** of H to be

$$N(H) = \{x \in G : x^{-1}Hx = H\}.$$

(a) **(18 points)** Prove that $N(H)$ is a subgroup of G.

(b) **(7 points)** It is a fact, which you may assume, that H is a subgroup of $N(H)$. Prove that H is a **normal** subgroup of $N(H)$.

Proof. **(a)**: (Nonempty): We have $e \in N(H)$, since $e^{-1}He = eH = H$.

(Closure): Given $x, y \in N(H)$, we have

$$(xy)^{-1}H(xy) = y^{-1}x^{-1}Hxy = y^{-1}(x^{-1}Hx)y = y^{-1}Hy = H,$$

so $xy \in H$.

(Inverses): Given $x \in H$, we have $x^{-1}Hx = H$. Multiplying by x on the left and x^{-1} on the right, then, we get $H = xHx^{-1}$. Thus,

$$(x^{-1})^{-1}Hx^{-1} = xHx^{-1} = H,$$

and so $x^{-1} \in H$.

Thus, $N(H)$ is a subgroup of G QED (a)

(b): We are told H is a subgroup of $N(H)$.

Given $x \in N(H)$, we have $x^{-1}Hx = H$, exactly the condition we need. Thus, H is indeed a *normal* subgroup of $N(H)$. QED (b)

2. **(25 points)** Let G_1, G_2 be groups, let $H_2 \subseteq G_2$ be a subgroup, and let $\phi : G_1 \to G_2$ be a homomorphism. Define

$$H_1 = \{x \in G_1 \ : \ \phi(x) \in H_2\}.$$

It is a fact, which you may assume, that H_1 is a subgroup of G_1.

Prove that for any $x, y \in G_1$, $H_1x = H_1y$ **if and only if** $H_2\phi(x) = H_2\phi(y)$.

Proof. ($\Longrightarrow$) Given $x, y \in G_1$ such that $H_1x = H_1y$, we have $xy^{-1} \in H_1$.

Thus, $\phi(xy^{-1}) \in H_2$. Since ϕ is a homomorphism, we have $\phi(x)\phi(y)^{-1} \in H_2$.

That is, $H_2\phi(x) = H_2\phi(y)$.

($\Longleftarrow$) Given $x, y \in G_1$ such that $H_2\phi(x) = H_2\phi(y)$, we have $\phi(x)\phi(y)^{-1} \in H_2$.

That is, $\phi(xy^{-1}) \in H_2$, since ϕ is a homomorphism.

Thus, $xy^{-1} \in H_1$, i.e., $H_1x = H_1y$. QED

3. **(25 points)** Consider the group S_6 of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Let $\sigma \in S_6$ be the permutation

$$\sigma = (3\ 6\ 2)(1\ 6\ 3)(1\ 4\ 2\ 5)(4\ 6).$$

(a) **(9 points)** Write σ as a product of **disjoint** cycles.

(b) **(7 points)** Compute the **order** of σ.

(c) **(9 points)** For each $i = 1, 2, 3, 4, 5$, let $\tau_i = (i\ 6)\sigma$.
For each such i, determine whether τ_i is σ **even** or **odd**.

Answers. (a): $\sigma = (1\ 4\ 6\ 3)(2\ 5)$

(b): $o(\sigma) = \text{lcm}(4, 2) = 4$.

(c): σ is the product of a 4-cycle (odd) and 2-cycle (also odd), so σ itself is even. Meanwhile, each $(i\ 6)$ is a 2-cycle and hence odd. So each τ_i is an odd composed with an even and hence is **odd**.

4. **(25 points)** Let R be a ring.

(a) **(6 points)** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

(b) **(19 points)** Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ from the real line to itself, under the usual operations of multiplication and addition of functions. (You may assume that R **is** indeed a ring under these operations.) Let

$$I = \{f \in R : f(3) = f(5) = 0\}.$$

Prove that I is an ideal of R.

Answer/Proof. (a): To say $I \subseteq R$ is an ideal of R means:

- I is nonempty,
- for all $x, y \in I$, we have $x + y \in I$,
- for all $x \in I$, we have $-x \in I$,
- for all $x \in I$ and $a \in R$, we have $ax, xa \in I$.

(b): **Proof**. (Nonempty): We have $0_R \in I$ since $0_R(3) = 0$ and $0_R(5) = 0$.
(Closed under +): Given $f, g \in I$, we have

$$(f + g)(3) = f(3) + g(3) = 0 + 0 = 0 \quad \text{and} \quad (f + g)(5) = f(3) + g(5) = 0 + 0 = 0,$$

so $f + g \in I$.

(Closed under $-$): Given $f \in I$, we have

$$(-f)(3) = -f(3) = -0 = 0 \quad \text{and} \quad (-f)(5) = -f(5) = -0 = 0,$$

so $-f \in I$.
(Ideal property): Given $f \in I$ and $g \in R$, we have

$$(fg)(3) = f(3)g(3) = 0 \cdot g(3) = 0 \quad \text{and} \quad (fg)(5) = f(5)g(5) = 0 \cdot g(5) = 0,$$

so $fg \in I$. We also have

$$(gf)(3) = g(3)f(3) = g(3) \cdot 0 = 0 \quad \text{and} \quad (gf)(5) = g(5)f(5) = g(5) \cdot 0 = 0,$$

so $gf \in I$. QED (b)