Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

February 2019

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. [25 points] Let G be a group, let $H \subseteq G$ be a subgroup, and define the **normalizer** of H to be
$$N(H) = \{x \in G : x^{-1}Hx = H\}.$$
   (a) [18 points] Prove that $N(H)$ is a subgroup of G.
   (b) [7 points] It is a fact, which you may assume, that H is a subgroup of $N(H)$. Prove that H is a **normal** subgroup of $N(H)$.

2. [25 points] Let G_1, G_2 be groups, let $H_2 \subseteq G_2$ be a subgroup, and let $\phi : G_1 \to G_2$ be a homomorphism. Define
$$H_1 = \{x \in G_1 \, : \, \phi(x) \in H_2\}.$$
It is a fact, which you may assume, that H_1 is a subgroup of G_1.
Prove that for any $x, y \in G_1$, $H_1x = H_1y$ **if and only if** $H_2\phi(x) = H_2\phi(y)$.

3. [25 points] Consider the group S_6 of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Let $\sigma \in S_6$ be the permutation
$$\sigma = (3\ 6\ 2)(1\ 6\ 3)(1\ 4\ 2\ 5)(4\ 6).$$
   (a) [9 points] Write σ as a product of **disjoint** cycles.
   (b) [7 points] Compute the **order** of σ.
   (c) [9 points] For each $i = 1, 2, 3, 4, 5$, let $\tau_i = (i\ 6)\sigma$. For each such i, determine whether τ_i is σ **even** or **odd**.

4. [25 points] Let R be a ring.
   (a) [6 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.
   (b) [19 points] Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ from the real line to itself, under the usual operations of multiplication and addition of functions. (You may assume that R **is** indeed a ring under these operations.) Let
$$I = \{f \in R : f(3) = f(5) = 0\}.$$
   Prove that I is an ideal of R.