Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

January 2019

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. (a) [5 points] Let U be a subset of the real numbers $\mathbf{R}$. State the definition of what it means for U to be an open set.

   Solution: A set U is open if for every $x \in U$ there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq U$. Here $V_\epsilon(x) = (x - \epsilon, x + \epsilon)$.

   (b) [10 points] Suppose that $U_1, U_2, \ldots, U_n$ are open subsets of $\mathbf{R}$. Using your definition in part (a), prove that the intersection of these open sets is open; namely,

$$U = \bigcap_{i=1}^{n} U_i \text{ is open}$$

   Solution: Let $x \in U = \bigcap_{i=1}^{n} U_i$. Since $x \in U_i$ and U_i is open for every $1 \leq i \leq n$, we know that for each i there exists an $\epsilon_i > 0$ such that $V_{\epsilon_i}(x) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Since $\{\epsilon_1, \ldots, \epsilon_n\}$ is a finite set of positive real numbers we have that $\epsilon > 0$ and by construction for each $1 \leq i \leq n$ we have $V_\epsilon(x) \subseteq V_{\epsilon_i}(x) \subseteq U_i$. Therefore, by the definition of intersection we have that $V_\epsilon(x) \subseteq U = \bigcap_{i=1}^{n} U_i$ and thus U is open.

   (c) [10 points] Give an example which shows that the intersection of an infinite number of open sets in $\mathbf{R}$ may not be open.

   Solution: Using the Archimedean property we can show that

$$\bigcap_{i=1}^{\infty} \left(0, 1 + \frac{1}{i}\right) = (0, 1].$$

2. (a) [5 points] Complete the following definition: A sequence of real numbers $\{a_n\}$ *converges* to the limit L if ...

   Solution: for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - L| < \epsilon$.

   (b) [5 points] Complete the following definition: A sequence of real numbers $\{a_n\}$ is *Cauchy* if ...

   Solution: for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_m - a_n| < \epsilon$.

   (c) [15 points] Prove that if the sequence $\{a_n\}$ converges (to L say), then $\{a_n\}$ is Cauchy.

   Solution: Suppose that $\{a_n\}$ converges to $L \in \mathbf{R}$ and let $\epsilon > 0$ be given. Since $a_n \to L$ there exists and $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that

$$|a_n - L| < \frac{\epsilon}{2}.$$

   Therefore, whenever $m, n \geq N$ it follows that

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

   Thus, $\{a_n\}$ is Cauchy.

3. (a) [5 points] State the Intermediate Value Theorem.

   Solution: Let $f\colon [1,b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists a $c \in (a,b)$ such that $f(c) = L$.

   (b) [10 points] Prove that the polynomial $f(x) = x^3 - 3x^2 + 1$ has at least one root. Recall that a root is a real number z such that $f(z) = 0$.

   Solution: Notice that

$$f(-1) = (-1)^3 - 3(-1)^2 + 1 = -3, \text{ and}$$
$$f(0) = 0^3 - 3(0) + 1 = 1.$$

   Since f is a polynomial it is continuous on $[-1,0]$ and $f(-1) = -3 < 0 < 1 = f(0)$. Therefore by the Intermediate Value Theorem there exits a real number $c \in (-1,0)$ such that $f(c) = 0$.

   (c) [10 points] Prove that $f(x) = x^3 - 3x^2 + 1$ has three real roots. (You may assume that f has no more than three (real) roots.)

   Solution: Notice that

$$f(-1) = (-1)^3 - 3(-1)^2 + 1 = -3, f(0) \qquad = 0^3 - 3(0) + 1 = 1,$$
$$f(1) = (1)^3 - 3(1)^2 + 1 = -1, \text{ and}$$
$$f(3) = (3)^3 - 3(3)^2 + 1 = 1.$$

   Since f is a polynomial, we know it is actually continuous on the real line and so we can apply the Intermediate Value Theorem to f on the intervals $[-1,0]$, $[0,1]$, and $[1,3]$ we get that there are real numbers $c_1 \in (-1,0)$, $c_2 \in (0,1)$, and $c_3 \in (1,3)$ such that $f(c_1) = f(c_2) = f(c_3) = 0$.

   Thus, f has at least 3 real roots and we know it can't have any more than 3 roots. So f has exactly 3 real roots.

4. (a) [5 points] State the Heine-Borel Theorem.

   Solution: Let $K \subseteq \mathbf{R}$. The following are equivalent:

   i) The set K is a compact set. I.e., every sequence contained in K has a convergent subsequence that converges in K.

   ii) The set K is closed and bounded

   iii) Every open cover of K has a finite subcover.

   (b) [20 points] Suppose that K is a compact subset of $\mathbf{R}$ and that $f\colon K \longrightarrow \mathbf{R}$ is a continuous function. Under these assumptions, pick one of the following two results and prove it. Please clearly state which result you're aiming to prove.

   (i) The image of K under f, $f(K) = \{f(x) \in \mathbf{R} \mid x \in K\}$, is compact.

   Solution: Let $\{y_n\}$ be a sequence such that for all $n \in \mathbf{N}$, we have $y_n \in f(K)$. By the definition of $f(K)$ we have that for each $n \in \mathbf{N}$ there exists an $x_n \in K$

such that $f(x_n) = y_n$. Since K is compact and $\{x_n\}$ is a sequence contained in K we know that there is a subsequence x_{n_k} that converges in K. Suppose that $\lim x_{n_k} = \alpha \in K$. Since f is continuous on K and since $\{x_n\}$ and α are contained in K, we have that $\lim f(x_{n_k}) = f(\alpha) \in f(K)$. Therefore $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$ that converges in $f(k)$. So $f(K)$ is compact by definition.

(ii) f is uniformly continuous.

Solution: Suppose towards a contradiction that f is not uniformly continuous on K. Then there exists sequences $\{x_n\}$ and $\{y_n\}$ in K and an $\epsilon_0 > 0$ such that $\lim |x_n - y_n| = 0$ and for all $n \in \mathbf{N}$, $|f(x_n) - f(y_n)| \geq \epsilon_0$. Since K is compact there is a subsequence of $\{x_n\}$, call it $\{x_{n_k}\}$, that converges to some $x \in K$. Using the algebraic limit theorem we have that the corresponding subsequence $\{y_{n_k}\}$ also converges to $x \in K$ since

$$\lim y_{n_k} = \lim(y_{n_k} - x_{n_k}) + x_{n_k} = \lim(y_{n_k} - x_{n_k}) + \lim x_{n_k} = 0 + x = x.$$

Next, because f is continuous on K and $\{x_{n_k}\}$, $\{y_{n_k}\}$, and x are all in K we have that $\lim f(x_{n_k}) = f(x)$ and $\lim f(y_{n_k}) = f(x)$. Again using the algebraic limit theorem we have

$$\lim(f(x_{n_k}) - f(y_{n_k})) = f(x) - f(x) = 0$$

contradicting the assumption that for all $n \in \mathbf{N}$, $|f(x_n) - f(y_n)| \geq \epsilon_0$.