Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

February 2019

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. (a) [5 points] Let U be a subset of the real numbers $\mathbf{R}$. State the definition of what it means for U to be an open set.

   (b) [10 points] Suppose that $U_1, U_2, \ldots, U_n$ are open subsets of $\mathbf{R}$. Using your definition in part (a), prove that the intersection of these open sets is open; namely,

$$U = \bigcap_{i=1}^{n} U_i \text{ is open}$$

   (c) [10 points] Give an example which shows that the intersection of an infinite number of open sets in $\mathbf{R}$ may not be open.

2. (a) [5 points] Complete the following definition: A sequence of real numbers $\{a_n\}$ *converges* to the limit L if ...

   (b) [5 points] Complete the following definition: A sequence of real numbers $\{a_n\}$ is *Cauchy* if ...

   (c) [15 points] Prove that if the sequence $\{a_n\}$ converges (to L say), then $\{a_n\}$ is Cauchy.

3. (a) [5 points] State the Intermediate Value Theorem.

   (b) [10 points] Prove that the polynomial $f(x) = x^3 - 3x^2 + 1$ has at least one root. Recall that a root is a real number z such that $f(z) = 0$.

   (c) [10 points] Prove that $f(x) = x^3 - 3x^2 + 1$ has three real roots. (You may assume that f has no more than three (real) roots.)

4. (a) [5 points] State the Heine-Borel Theorem.

   (b) [20 points] Suppose that K is a compact subset of $\mathbf{R}$ and that $f\colon K \longrightarrow \mathbf{R}$ is a continuous function. Under these assumptions, pick ONE and ONLY ONE of the following two results and prove it. Please clearly state which result you're aiming to prove.
   (i) The image of K under f, $f(K) = \{f(x) \in \mathbf{R} \mid x \in K\}$, is compact.
   (ii) f is uniformly continuous.