Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Multivariable Calculus and Linear Algebra ▷

January 2019

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. A fly is buzzing around a room in which the temperature is given in degrees Celsius by

$$T(x, y, z) = x^2 + y^2 + 3z^2 + 17.$$

Suppose that the fly is currently at the point $(1, 2, -1)$.

(a) [10 points] The fly wants to warm up. In what direction should it fly? Find a vector pointing in the direction in which the temperature increases most rapidly from the fly's current position.

(b) [15 points] Suppose the fly moves from the point $(1, 2, -1)$ in the direction of the vector $\langle 4, 0, 3\rangle$. Find the directional derivative of the temperature in that direction. Will the fly feel warmer or colder?

Solution: We need the gradient:

$$\nabla T(x, y, z) = \langle 2x, 2y, 6z\rangle.$$

$$\nabla T(1, 2, -1) = \langle 2, 4, -6\rangle.$$

(a) The temperature increases most rapidly in the direction of the gradient, $\nabla T(1, 2, -1) = \langle 2, 4, -6\rangle$. Any vector that is a positive scalar multiple of this vector is valid here.

(b) The directional derivative is

$$\begin{aligned} D_{\mathbf{u}}(1, 2, -1) &= \nabla T(1, 2, -1) \cdot \langle 4, 0, 3\rangle/5 \\ &= \langle 2, 4, -6\rangle \cdot \langle 4, 0, 3\rangle/5 \\ &= (8 + 0 - 18)/5 = -2, \end{aligned}$$

so decreasing (getting colder in that direction).

2. [25 points] Let $f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

Solution: Take the partial derivatives of $f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$ and set equal to zero to find the critical points. $f_y = -3x^2 - 6y = 0$ leads to $y = -x^2/2$. Substituting into $f_x = 6x^2 - 6xy - 24x = 0$ leads to

$$3x^3 + 6x^2 - 24x = 3x(x^2 + 2x - 8) = 3x(x - 2)(x + 4) = 0.$$

The solutions are $x = 0, 2, -4$, so the critical points are

$$(0, 0), (2, -2), (-4, -8)$$

Use the Second Partials Test to classify the points:

$D(0, 0) = 144 > 0$ and $f_{xx}(0, 0) = -24 < 0$, so $(0, 0)$ is a local maximum.

$D(2, -2) = -72 < 0$ so $(2, -2)$ is a saddle point.

$D(-4, -8) = -24 \cdot 18 < 0$ so $(-4, -8)$ is also a saddle point.

3. [25 points] Find the volume of the region that lies both inside the sphere $x^2+y^2+z^2=6$ and above the paraboloid $z=x^2+y^2$.

Solution: Working in polar or cylindrical coordinates is easiest for this problem. Find where the sphere $r^2+z^2=6$ intersects the paraboloid $z=r^2$ by finding the positive solution of $z^2+z-6=0$, which is $z=2$, corresponding to $r=\sqrt{2}$.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\sqrt{2}} (\sqrt{6-r^2}-r^2)r\,dr\,d\theta \\
&= 2\pi\left(-\frac{1}{3}(6-r^2)^{3/2}-\frac{1}{4}r^4\right)\Bigg|_0^{\sqrt{2}} = 2\pi(2\sqrt{6}-11/3)
\end{aligned}
$$

4. [25 points] Compute $\int_C x^2y\,dx+xy^2\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, traversed in the counterclockwise direction.

Solution: Use Green's Theorem to convert to easier double integral:

$$\int_C x^2y\,dx+xy^2\,dy = \int_0^1\int_0^x (y^2-x^2)\,dy\,dx = -\frac{2}{3}\int_0^1 x^3\,dx = -1/6$$

5. [25 points] Let V be a vector space and let $\alpha=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ and $\beta=\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ each be a basis for V. Suppose that

$$
\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 \\
\mathbf{w}_2 &= \mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3 \\
\mathbf{w}_3 &= 2\mathbf{v}_1 + 5\mathbf{v}_3.
\end{aligned}
$$

Let $\mathbf{v}\in V$ such that $\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3$ for some $a_1,a_2,a_3\in\mathbb{R}$. Find $b_1,b_2,b_3\in\mathbb{R}$ (in terms of a_1,a_2, and a_3) such that $\mathbf{v}=b_1\mathbf{w}_1+b_2\mathbf{w}_2+b_3\mathbf{w}_3$.

Solution: From the given information we have

$$[I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 3 & 5 \end{bmatrix}.$$

Further

$$[I]_\alpha^\beta = \left([I]_\alpha^\beta\right)^{-1} = \begin{bmatrix} 5 & 1 & -2 \\ -10 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix}$$

Thus, if $\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+a_3\mathbf{v}_3$, then $[\mathbf{v}]_\alpha = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\mathbf{v}]_\beta = [I]_\alpha^\beta[\mathbf{v}]_\alpha = \begin{bmatrix} 5 & 1 & -2 \\ -10 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5a_1+a_2-2a_3 \\ -10a_1-a_2+4a_3 \\ 3a_1-a_3 \end{bmatrix}.$$

So

$$\begin{aligned} b_1 &= 5a_1 + a_2 - 2a_3 \\ b_2 &= -10a_1 - a_2 + 4a_3 \\ b_3 &= 3a_1 - a_3. \end{aligned}$$

6. [25 points] Let V be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ be linearly independent. Prove that for any $\mathbf{v} \in V$ such that $\mathbf{v} \notin \text{Span}(S)$ the set $S \cup \{\mathbf{v}\}$ is also linearly independent.

   Solution: Suppose $a, a_1, \ldots, a_k \in \mathbb{R}$ such that

$$a\mathbf{v} + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}. \tag{1}$$

   We aim to show that $a = a_1 = \cdots = a_k = 0$.

   Case 1: $a = 0$.
   In this case, Equation (1) becomes

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0},$$

   and since S is linearly independent we have that $a_1 = \cdots = a_k = 0$ completing this case.

   Case 2: $a \neq 0$.
   In this case we can rewrite Equation (1) to be

$$\left(\frac{-a_1}{a}\right)\mathbf{v}_1 + \left(\frac{-a_2}{a}\right)\mathbf{v}_2 + \cdots + \left(\frac{-a_k}{a}\right)\mathbf{v}_k = \mathbf{v}. \tag{2}$$

   Equation 2 gives us that $\mathbf{v} \in \text{Span}(S)$ contradicting our assumption that $\mathbf{v} \notin \text{Span}(S)$ and completing this case.

7. Let $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2 and let $T : \mathbb{R}^3 \to P_2$ be the linear map given by

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 1 + x + x^2, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = x, \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = 1 + x^2.$$

   (a) [10 points] Find $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

   (b) [5 points] State the definition of an isomorphism between vector spaces.

   (c) [10 points] Is T an isomorphism? Justify your answer.

   Solution:

(a) Notice that

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = -\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Therefore,

$$\begin{aligned} T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) &= T\left(-\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \\ &= -T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + 3T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \\ &= -(1+x+x^2) - (x) + 3(1+x^2) = 2 - 2x + 2x^2. \end{aligned}$$

(b) A function $S : V \to W$ between two vector spaces is an isomorphism if it is a linear map that is both one-to-one and onto.

(c) The map above is not an isomorphism since it is not one-to-one (or onto). Notice that

$$\begin{aligned} T\left(\begin{bmatrix}0\\-1\\-1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \\ &= (1+x+x^2) - (1+x^2) = x = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right), \end{aligned}$$

and $\begin{bmatrix}0\\-1\\-1\end{bmatrix} \neq \begin{bmatrix}1\\1\\1\end{bmatrix}$.

8. Let $A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}$.

(a) [10 points] Find all the eigenvalues of A.

(b) [15 points] If possible, find a basis for $\mathbb{R}^3$ consisting only of eigenvectors of A. If this is not possible, explain why.

Solution:

(a) We start by computing the characteristic polynomial of A.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix}1-\lambda & 0 & 1\\0 & 1-\lambda & 0\\1 & 0 & 1-\lambda\end{vmatrix} = (1-\lambda)\begin{vmatrix}1-\lambda & 1\\1 & 1-\lambda\end{vmatrix} \\ &= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = -\lambda(1-\lambda)(2-\lambda). \end{aligned}$$

From this we see that the eigenvalues are $\lambda = 0, 1, 2$ since these are the roots of the characteristic polynomial.

(b) Here we compute a basis for each eigenspace by solving the homogenous system of equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for each eigenvalue λ. Doing this yields

$$E_0 = \text{Span}\left(\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}\right), E_1 = \text{Span}\left(\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}\right), E_2 = \text{Span}\left(\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\right).$$

Therefore,

$$\alpha = \left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

is a basis for $\mathbb{R}^3$ that is made of eigenvectors of A.