Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Multivariable Calculus and Linear Algebra ▷

February 2019

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or oFebruaryher aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: __________________

Grader #2: __________________

1. A fly is buzzing around a room in which the temperature is given in degrees Celsius by

$$T(x, y, z) = x^2 + y^2 + 3z^2 + 17.$$

Suppose that the fly is currently at the point $(1, 2, -1)$.

(a) [10 points] The fly wants to warm up. In what direction should it fly? Find a vector pointing in the direction in which the temperature increases most rapidly from the fly's current position.

(b) [15 points] Suppose the fly moves from the point $(1, 2, -1)$ in the direction of the vector $\langle 4, 0, 3\rangle$. Find the directional derivative of the temperature in that direction. Will the fly feel warmer or colder?

2. [25 points] Let $f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$. Find all critical points of f, and classify each as a local maximum, local minimum, or saddle point.

3. [25 points] Find the volume of the region that lies both inside the sphere $x^2+y^2+z^2 = 6$ and above the paraboloid $z = x^2 + y^2$.

4. [25 points] Compute $\displaystyle\int_C x^2y\,dx + xy^2\,dy$ where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$, traversed in the counterclockwise direction.

5. [25 points] Let V be a vector space and let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ each be a basis for V. Suppose that

$$\begin{aligned}\mathbf{w}_1 &= \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3\\ \mathbf{w}_2 &= \mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3\\ \mathbf{w}_3 &= 2\mathbf{v}_1 \qquad + 5\mathbf{v}_3.\end{aligned}$$

Let $\mathbf{v} \in V$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$. Find $b_1, b_2, b_3 \in \mathbb{R}$ (in terms of a_1, a_2, and a_3) such that $\mathbf{v} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3$.

6. [25 points] Let V be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ be linearly independent. Prove that for any $\mathbf{v} \in V$ such that $\mathbf{v} \notin \operatorname{Span}(S)$ the set $S \cup \{\mathbf{v}\}$ is also linearly independent.

7. Let $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2 and let $T : \mathbb{R}^3 \to P_2$ be the linear map given by

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 1 + x + x^2, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = x, \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = 1 + x^2.$$

(a) [10 points] Find $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

(b) [5 points] State the definition of an isomorphism between vector spaces.

(c) [10 points] Is T an isomorphism? Justify your answer.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) [10 points] Find all the eigenvalues of A.

   (b) [15 points] If possible, find a basis for $\mathbb{R}^3$ consisting only of eigenvectors of A. If this is not possible, explain why.