Amherst College Department of Mathematics and Statistics

Solutions to Comprehensive Examination: Algebra

January, 2020

1. **(25 points)** Let G, H be groups, and let $\phi, \psi : G \to H$ be homomorphisms. Define

$$E = \{x \in G \,|\, \phi(x) = \psi(x)\}.$$

Prove that E is a subgroup of G.

Proof. (Nonempty): Let e_G and e_H be the identity elements of G and H. By properties of homomorphisms, we have $\phi(e_G) = e_H = \psi(e_G)$. Thus, $e_G \in E$, and hence $E \neq \varnothing$.

(Closure): Given $x, y \in E$, we have

$$\phi(xy) = \phi(x)\phi(y) = \psi(x)\psi(y) = \psi(xy),$$

where the first and third equalities are by definition of homomorphism, and the second is because $x, y \in E$. Thus, we have $xy \in E$.

(Inverses): Given $x \in E$, we have

$$\phi(x^{-1}) = \phi(x)^{-1} = \psi(x)^{-1} = \psi(x^{-1}),$$

where the first and third equalities are by properties of homomorphisms, and the second is because $x \in E$. Thus, we have $x^{-1} \in E$.

Therefore, E is a subgroup of G. QED

2. **(25 points)** Let G be an abelian group, and define

$$T = \{g \in G \,|\, g \text{ has finite order}\}.$$

It is a fact, which you may assume, that T is a normal subgroup of G. Prove that the only element of the quotient group G/T that has finite order is the identity element.

Proof. Given $Tg \in G/T$ of finite order, we must show that $Tg = Te$.

By assumption, there exists $n \geq 1$ such that $(Tg)^n = Te$. Thus, $T(g^n) = Te$, so by the coset criterion, $g^n e^{-1} \in T$, i.e., $g^n \in T$.

Since $g^n \in T$, there is some $m \geq 1$ such that $(g^n)^m \in T$. Therefore, $g^{nm} = e$. Since $nm \geq 1$, it follows that g has finite order, and hence $g \in T$. Thus, $ge^{-1} = T$, and hence $Tg = Te$, by the coset criterion. QED

3. **(25 points)** Consider the group S_9 of permutations of the set $\{1, 2, 3, \ldots, 9\}$. Let $\sigma, \tau \in S_9$ be the permutations

$$\sigma = (1, 2, 3, 4)(5, 6) \quad \text{and} \quad \tau = (1, 6, 8)(2, 7, 3, 5, 4).$$

(a) **(8 points)** Write $\sigma\tau$ as a product of **disjoint** cycles.

(b) **(8 points)** Compute the **order** of each of σ, τ, and $\sigma\tau$.

(c) **(9 points)** Decide whether each of σ, τ, and $\sigma\tau$ is an **even** or **odd** permutation; don't forget to justify.

Answers. (a): $\sigma\tau = (1\ 5)(2\ 7\ 4\ 3\ 6\ 8)$
[Alternatively, if a student chooses to read the cycles as composing from left to right (which is highly nonstandard but internally consistent), the answer would be $(1\ 7\ 3\ 2\ 5\ 8)(4\ 6)$.]

(b): $o(\sigma) = \text{lcm}(4, 2) = 4$.
$o(\tau) = \text{lcm}(3, 5) = 15$.
$o(\sigma\tau) = \text{lcm}(2, 6) = 6$.

(c): σ is the product of a 4-cycle (odd) and 2-cycle (also odd), so σ is **even**.
τ is the product of a 3-cycle (even) and a 5-cycle (also even), so τ is also **even**.
$\sigma\tau$ is a product of σ (even) and τ (even), so it is **even**.
[Alternatively, $\sigma\tau$ is the product of a 2-cycle (odd) and a 6-cycle (also odd), so $\sigma\tau$ is even.]

4. **(25 points)** Let R be a ring.

(a) **(6 points)** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

(b) **(19 points)** Suppose R is commutative, and let $S \subseteq R$ be a subset of R. Define the *annihilator* of S in R to be

$$\text{Ann}(S) = \{x \in R : xs = 0 \text{ for every } s \in S\}.$$

Prove that $\text{Ann}(S)$ is an ideal of R.

Answer/Proof. (a): To say $I \subseteq R$ is an ideal of R means:

- I is nonempty,
- for all $x, y \in I$, we have $x - y \in I$,
- for all $x \in I$ and $a \in R$, we have $ax, xa \in I$.

(b): **Proof**. Let $I = \text{Ann}(S)$.
(Nonempty): We claim $0 \in I$. Indeed, given $s \in S$, we have $0s = 0$, so $0 \in I$.
(Closed under $-$): Given $x, y \in I$, and given $s \in S$, we have

$$(x - y)s = xs - ys = 0 - 0 = 0,$$

so $x - y \in I$.

(Ideal property): Given $x \in I$ and $a \in R$, and given $s \in S$, we have

$$(ax)s = a(xs) = a(0) = 0, \quad \text{and} \quad (xa)s = (ax)s = 0.$$

Thus, $ax, xa \in I$. QED (b)