Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

January 2020

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. [25 points] Let G, H be groups, and let $\phi, \psi : G \to H$ be homomorphisms. Define

$$E = \{x \in G \mid \phi(x) = \psi(x)\}.$$

Prove that E is a subgroup of G.

2. [25 points] Let G be an abelian group, and define

$$T = \{g \in G \mid g \text{ has finite order}\}.$$

It is a fact, which you may assume, that T is a normal subgroup of G. Prove that the only element of the quotient group G/T that has finite order is the identity element.

3. [25 points] Consider the group S_9 of permutations of the set $\{1, 2, 3, \ldots, 9\}$. Let $\sigma, \tau \in S_9$ be the permutations

$$\sigma = (1, 2, 3, 4)(5, 6) \quad \text{and} \quad \tau = (1, 6, 8)(2, 7, 3, 5, 4).$$

(a) [8 points] Write $\sigma\tau$ as a product of **disjoint** cycles.

(b) [8 points] Compute the **order** of each of σ, τ, and $\sigma\tau$.

(c) [9 points] Decide whether each of σ, τ, and $\sigma\tau$ is an **even** or **odd** permutation; don't forget to justify.

4. [25 points] Let R be a ring.

(a) [6 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

(b) [19 points] Suppose R is commutative, and let $S \subseteq R$ be a subset of R. Define the *annihilator* of S in R to be

$$\text{Ann}(S) = \{x \in R : xs = 0 \text{ for every } s \in S\}.$$

Prove that $\text{Ann}(S)$ is an ideal of R.