Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

January 2020

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.
- Throughout this exam $\mathbf{R}$ will denote the real number system.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. (a) [5 points] State the Axiom of Completeness for $\mathbf{R}$.

Solution: Every nonempty subset of $\mathbf{R}$ that is bounded from above (in $\mathbf{R}$) has a least upper bound that belongs to $\mathbf{R}$. □

(b) [20 points] Let $A \subseteq \mathbf{R}$ be a nonempty set which is bounded from below and define

$$B := \{-a : a \in A\}.$$

Prove that $\sup(B) \in \mathbf{R}$ exists and $\inf(A) = -\sup(B)$.

Proof. We begin by showing that $\sup(B) \in \mathbf{R}$. Note that $B \subseteq \mathbf{R}$ is nonempty since A is nonempty. Additionally, given that A is bounded from below we can find $\ell \in \mathbf{R}$ such that $\ell \leq a$ for all $a \in A$. Then $-\ell \in \mathbf{R}$ satisfies $-\ell \geq -a$ for all $a \in A$ which implies B is bounded from above by $-\ell$. Thus, $\sup(B) \in \mathbf{R}$ by the Axiom of Completeness.

Moving on, to show that $\inf(A) = -\sup(B)$, we first need to show that $-\sup(B)$ is a lower bound for A. Since $\sup(B) \geq -a$ for all $a \in A$ we have $-\sup(B) \leq a$ for all $a \in A$. Hence, $-\sup(B)$ is a lower bound for A, as wanted. Next, suppose that $\alpha \in \mathbf{R}$ is any lower bound for A, i.e., suppose $\alpha \leq a$ for all $a \in A$. Then, $-\alpha \geq -a$ for all $a \in A$ which implies that $-\alpha \in \mathbf{R}$ is an upper bound for B. It follows that $\alpha \geq -\sup(B)$. Hence, $-\sup(B)$ is the greatest lower bound for A, i.e., $\inf(A) = -\sup(B)$. □

2. (a) [5 points] State the Mean Value Theorem.

Solution: Let $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f\colon [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists at least one point $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$. □

(b) [20 points] Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at every point in $\mathbf{R}$. Use part (a) to prove that if $f'(x) = 0$ for all $x \in \mathbf{R}$ then $f(x) = k$ for some constant $k \in \mathbf{R}$.

Solution: To show that f is a constant function, we will show that $f(a) = f(b)$ for all $a, b \in \mathbf{R}$, $a < b$. Fix $a, b \in \mathbf{R}$, $a < b$. Since f is differentiable at every point in $\mathbf{R}$ it is also continuous at every point in $\mathbf{R}$. It follows by definition that f is continuous on $[a,b]$ and differentiable on (a,b). In light of the Mean Value Theorem, we can find $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$. By assumption, we have $f'(c) = 0$. Hence, $f(b) - f(a) = f'(c)(b-a) = 0$ which gives $f(a) = f(b)$, as wanted. □

3. For each $n \in \mathbb{N}$, let $f_n\colon (0,\infty) \to \mathbf{R}$ be given by $f_n(x) := \dfrac{1}{n^2 x}$ for all $x \in (0,\infty)$.

(a) [10 points] Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise on $(0,\infty)$. Be sure to define the limit function $f\colon (0,\infty) \to \mathbf{R}$.

Proof. Define $f\colon (0,\infty) \to \mathbf{R}$ by setting $f(x) := 0$ for all $x \in (0,\infty)$. Fix $a \in (0,\infty)$. Then by the Algebraic Limit Theorem for sequences we have

$$\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \frac{1}{n^2 a} = \frac{1}{a}\left(\lim_{n\to\infty} \frac{1}{n^2}\right) = 0 = f(a).$$

Since $a \in (0,\infty)$ was chosen arbitrarily, we have that $(f_n)_{n=1}^{\infty}$ converges pointwise on $(0,\infty)$ to f. $\square$

(b) [15 points] Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[c,\infty)$ for every $c \in (0,\infty)$.

Proof. Fix $c \in (0,\infty)$, along with an arbitrary $\varepsilon \in (0,\infty)$. By the Archimedean Property, there exists $N \in \mathbf{N}$ with $N > \frac{1}{\sqrt{c\varepsilon}}$. Then if $n \in \mathbf{N}$ satisfies $n \geq N$ and $x \in [c,\infty)$, then

$$|f_n(x) - f(x)| = \left|\frac{1}{n^2 x} - 0\right| = \frac{1}{n^2 x} \leq \frac{1}{N^2 c} < \varepsilon.$$

It follows by definition that $(f_n)_{n=1}^{\infty}$ converges uniformly on $[c,\infty)$ to f. $\square$

4. (a) [5 points] Finish the following definition: *A sequence* $(a_n)_{n=1}^{\infty}$ *of real numbers is said to be* ***Cauchy*** *if...*

   Solution: for every $\varepsilon \in (0, \infty)$ there exists $N \in \mathbf{N}$ such that for all $n, m \in \mathbf{N}$ with $n, m \geq N$, one has $|a_n - a_m| < \varepsilon$. $\square$

   (b) [5 points] Finish the following definition: *A function* $f\colon \mathbf{R} \to \mathbf{R}$ *is said to* ***uniformly continuous on a nonempty set*** $E \subseteq \mathbf{R}$ *if...*

   Solution: for every $\varepsilon \in (0, \infty)$ there exists $\delta \in (0, \infty)$ such that for all $x, y \in E$ with $|x - y| < \delta$, one has $|f(x) - f(y)| < \varepsilon$. $\square$

   (c) [15 points] Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a uniformly continuous on a nonempty set $E \subseteq \mathbf{R}$. Prove that if $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of points in E then the sequence $\big(f(a_n)\big)_{n=1}^{\infty}$ is Cauchy.

   Solution: Suppose $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of points in E. Fix $\varepsilon \in (0, \infty)$. Since f is uniformly continuous on E we have that there exists $\delta \in (0, \infty)$ such that for all $x, y \in E$ with $|x - y| < \delta$ one has $|f(x) - f(y)| < \varepsilon$. Relying on the assumption that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and the fact that $\delta \in (0, \infty)$, we are guaranteed a number $N \in \mathbf{N}$ such that for all $n, m \in \mathbf{N}$ with $n, m \geq N$, one has $|a_n - a_m| < \delta$. Combining these observations, we can conclude that if $n, m \in \mathbf{N}$ with $n, m \geq N$ then $|f(a_n) - f(a_m)| < \varepsilon$. $\square$