Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Analysis ▷

January 2020

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.
- Throughout this exam $\mathbf{R}$ will denote the real number system.

For Department Use Only:

Grader #1: ________________

Grader #2: ________________

1. (a) [5 points] State the Axiom of Completeness for $\mathbf{R}$.

   (b) [20 points] Let $A \subseteq \mathbf{R}$ be a nonempty set which is bounded from below and define
$$B := \{-a : a \in A\}.$$
   Prove that $\sup(B) \in \mathbf{R}$ exists and $\inf(A) = -\sup(B)$.

2. (a) [5 points] State the Mean Value Theorem.

   (b) [20 points] Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at every point in $\mathbf{R}$. Use part (a) to prove that if $f'(x) = 0$ for all $x \in \mathbf{R}$ then $f(x) = k$ for some constant $k \in \mathbf{R}$.

3. For each $n \in \mathbb{N}$, let $f_n\colon (0,\infty) \to \mathbf{R}$ be given by $f_n(x) := \dfrac{1}{n^2 x}$ for all $x \in (0,\infty)$.

   (a) [10 points] Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise on $(0,\infty)$. Be sure to define the limit function $f\colon (0,\infty) \to \mathbf{R}$.

   (b) [15 points] Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on $[c,\infty)$ for every $c \in (0,\infty)$.

4. (a) [5 points] Finish the following definition: *A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is said to be* ***Cauchy*** *if...*

   (b) [5 points] Finish the following definition: *A function $f\colon \mathbf{R} \to \mathbf{R}$ is said to be* ***uniformly continuous on a nonempty set*** *$E \subseteq \mathbf{R}$ if...*

   (c) [15 points] Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a uniformly continuous on a nonempty set $E \subseteq \mathbf{R}$. Prove that if $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of points in E then the sequence $\big(f(a_n)\big)_{n=1}^{\infty}$ is Cauchy.