Multivariable Calc Problems for Math Comps 2020

Solutions

1. We need the gradient:

$$\nabla f(x,y) = \langle -x, -y \rangle / \sqrt{x^2+y^2}.$$

$$\nabla f(6,8) = \langle -3/5, -4/5 \rangle.$$

The directional derivative is then (turning the direction vector into a unit vector)

$$\langle -3/5, -4/5 \rangle \cdot \langle 3/5, -4/5 \rangle = -9/25 + 16/25 = 7/25$$

So the hiker is going uphill, experiencing a slope of 7/25.

2. Apply Lagrange multipliers, which yields the following system of equations to solve:

$$y = 8\lambda x$$
$$x = 18\lambda y$$
$$4x^2 + 9y^2 = 32$$

Substituting the first equation into the second gives $x = 144\lambda^2 x$, so either $x = 0$ (but $(0,0)$ doesn't satisfy the constraint) or $\lambda = \pm 1/12$. Plugging the latter into first equation gives $y = \pm 2/3x$; feeding that into the constraint leads to $4x^2 + 9 \cdot \frac{4}{9}x^2 = 32$, which simplifies to $x^2 = 4$. Hence the candidates are $f(\pm 2, \pm 4/3) = 8/3$ and $f(\pm 2, \mp 4/3) = -8/3$. We conclude that the absolute max of 8/3 occurs at $\pm(2, 4/3)$ and the absolute min of -8/3 occurs at $\pm(2, -4/3)$.

3. Working in either polar or spherical coordinates works fine for this problem. To use polar, determine where the sphere $r^2+z^2 = 6$ intersects the cone $z = \sqrt{3}r$ by finding the positive solution of $r^2+3r^2 = 4$, which is $r = 1$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (\sqrt{4-r^2} - \sqrt{3}r) r \, dr \, d\theta \\ &= 2\pi \left(-\frac{1}{3}(4-r^2)^{3/2} - \frac{\sqrt{3}}{3} r^3 \right) \Bigg|_0^1 = \frac{8\pi}{3}(2-\sqrt{3}) \end{aligned}$$

In spherical coordinates, the integral is

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8\pi}{3}(2-\sqrt{3})$$

4. Use Green's Theorem to convert to easier double integral:

$$\begin{aligned} \int_C (\ln x + y) \, dx - x^2 \, dy &= \int_1^3 \int_1^4 (-2x-1) \, dy \, dx \\ &= \int_1^3 3(-2x-1) \, dx \\ &= -3(x^2+x) \Big|_1^3 = -30 \end{aligned}$$

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5. Let $A = \begin{bmatrix} 1 & -2 & -3 & 3 \\ -1 & 2 & 0 & 3 \\ -1 & 2 & -2 & 7 \end{bmatrix}$.

   (a) (5 points) Explain what is meant by the *null space* (or *kernel*) of a matrix.

   (b) (10 points) Find a basis for the null space (kernel) of A.

   (c) (10 points) Find a basis for the column space (span of the columns) of A.

6. (25 points) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of a vector space V. Prove that $\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is also a basis for V.

7. Let V be a vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three vectors. Define $T:\ \mathbb{R}^3 \to V$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3.$$

   (a) (5 points) Prove that T is a *linear transformation.*

   (b) (10 points) Prove that if S is linearly independent, then T is one-to-one (injective).

   (c) (10 points) Prove that if $\text{Span}(S) = V$, then T is onto (surjective).

8. Let $A = \begin{bmatrix} -4 & 3 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

   (a) (10 points) Determine the eigenvalues of A.

   (b) (15 points) For a basis for the eigenspace of each eigenvalue.

Solutions

5. (a) The null space of A is the set

$$\{\mathbf{x} \in \mathbb{R}^4 : \; A\mathbf{x} = \mathbf{0}\}.$$

(b) The null space is unaffected by elementary row operations. Therefore we perform row-reduction as follows.

$$\begin{bmatrix} 1 & -2 & -3 & 3 \\ -1 & 2 & 0 & 3 \\ -1 & 2 & -2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 & 3 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -5 & 10 \end{bmatrix} \text{ (add R1 to both R2 and R3)}$$

$$\to \begin{bmatrix} 1 & -2 & -3 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -5 & 10 \end{bmatrix} \text{ (divide R2 by } -3\text{)}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (add 5R2 to R3 and 3R2 to R1)}$$

From the reduced row echelon form, me may directly read the general solution to $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} x_1 &= 2x_2 + 3x_4 \\ x_3 &= 2x_4 \\ x_2, x_4 & \quad \text{free variables.} \end{aligned}$$

In vector form, this reads

$$\begin{bmatrix} 2x_2 + 3x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore one basis for the null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(c) In the reduced row echelon form of A, pivots occur in columns 1 and 3. This mean that columns 1 and 3 of the original matrix necessarily form a basis for the column space. So one basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

6. First we claim that $\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is a linearly independent set. Suppose that $c_1, c_2, c_3 \in \mathbb{R}$ are constants such that

$$c_1(\mathbf{v}_1 - \mathbf{v}_3) + c_2(\mathbf{v}_2 - \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}.$$

Distributing and rearranging terms gives

$$(c_1 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + (-c_1 - c_2 + c_3)\mathbf{v}_3 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of V, it is linearly independent. Thus it follows from the equation above that the following three equations hold.

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ -c_1 - c_2 + c_3 &= 0 \end{aligned}$$

Adding all three equations together yields $3c_3 = 0$, hence $c_3 = 0$. Substituting this into the first two equations yields $c_1 = 0$ and $c_2 = 0$. So indeed $c_1 = c_2 = c_3 = 0$, so that original list $\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is linearly independent.

Now, $\dim V = 3$ since we are given that it has a basis of three elements. Hence any linearly independent set of three vectors in V must be a basis; it follows that $\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is a basis of V.

Alternatively, it is also possible to show directly that $\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ spans V.

7. (a) It suffices to show that for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and any $c \in \mathbb{R}$, $T(\mathbf{x} + c\mathbf{y}) = T(\mathbf{x}) + cT(\mathbf{y})$. We can compute as follows.

$$\begin{aligned} T(\mathbf{x} + c\mathbf{y}) &= T\left(\begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ x_3 + cy_3 \end{bmatrix}\right) \\ &= (x_1 + cy_1)\mathbf{v}_1 + (x_2 + cy_2)\mathbf{v}_2 + (x_3 + cy_3)\mathbf{v}_3 \\ &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) + c(y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3) \\ &= T(\mathbf{x}) + cT(\mathbf{y}), \end{aligned}$$

as desired.

(b) We recall that T is one-to-one if and only if it has trivial kernel. Therefore it suffices to show that $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Now, if $T(\mathbf{x}) = \mathbf{0}$, then

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0},$$

from which it follows, since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, that $x_1 = x_2 = x_3 = 0$, i.e. $\mathbf{x} = \mathbf{0}$. So the only element of the null space is $\mathbf{0}$, so indeed T is one-to-one.

(c) Suppose that $\mathbf{w}$ is any element of V. We must show that there exists $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{w}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V and $\mathbf{w} \in V$, there exist constants c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$. In other words

$$T\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = \mathbf{w}.$$

So indeed $\mathbf{w}$ is in the image of T. This shows that T is onto.

8. (a) The characteristic polynomial of A is

$$\begin{aligned}
\begin{vmatrix} -4-\lambda & 3 & 0 \\ -6 & 5-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} &= (-1-\lambda)\begin{vmatrix} -4-\lambda & 3 \\ -6 & 5-\lambda \end{vmatrix} \text{ (cofactor expansion)} \\
&= ((-4-\lambda)(5-\lambda) - 3(-6))\,(-1-\lambda) \\
&= (\lambda^2 - \lambda - 2)(-1-\lambda) \\
&= -(\lambda+1)(\lambda-2)(\lambda+1) \\
&= -(\lambda+1)^2(\lambda-2)
\end{aligned}$$

Hence there are two eigenvalues: $\lambda = -1$ and $\lambda = 2$.

(b) First consider $\lambda = -1$. Its eigenspace is the null space of

$$A + I = \begin{bmatrix} -3 & 3 & 0 \\ -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Canceling the second row with twice the first and dividing the first by (-3), we see that the reduced row echelon form of this matrix is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and thus its null space has basis $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. Hence this is the eigenspace of $\lambda = -1$.

Now consider $\lambda = 2$. Its eigenspace is the null space of

$$A - 2I = \begin{bmatrix} -6 & 3 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Canceling the second row, scaling the first and third row, and then reordering the row, we obtain the following reduced row echelon form of $A - 2I$:

$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The null space is therefore spanned by the vector $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$, or equivalently by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. So one basis for the eigenspace of $\lambda = 2$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$.