Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Multivariable Calculus and Linear Algebra ▷

January 2020

Number: ______

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: ____________________

Grader #2: ____________________

1. [25 points] Suppose a hiker is standing near the peak of a small mountain described by the function $f(x, y) = 5400 - \sqrt{x^2 + y^2}$ (all units in feet). If the hiker is currently at the point (6,8) and facing the direction given by the vector $\langle 3, -4 \rangle$, what slope is the hiker experiencing as they begin to take their next step? Are they going downhill or uphill?

2. [25 points] Find the absolute maximum and minimum values of the function
$$f(x, y) = xy$$
subject to the constraint $4x^2 + 9y^2 = 32$. Clearly state the points at which the extrema occur.

3. [25 points] Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{3x^2 + 3y^2}$.

4. [25 points] Compute $\int_C (\ln x + y)\, dx - x^2\, dy$ where C is the rectangle with vertices $(1, 1)$, $(3, 1)$, $(3, 4)$, and $(1, 4)$, traversed in the counterclockwise direction.

5. Let $A = \begin{bmatrix} 1 & -2 & -3 & 3 \\ -1 & 2 & 0 & 3 \\ -1 & 2 & -2 & 7 \end{bmatrix}$.
   (a) [5 points] Explain what is meant by the *null space* (or *kernel*) of a matrix.
   (b) [10 points] Find a basis for the null space (kernel) of A.
   (c) [10 points] Find a basis for the column space (span of the columns) of A.

6. [25 points] Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of a vector space V. Prove that
$$\{\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$$
is also a basis for V.

7. Let V be a vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three vectors. Define $T : \mathbb{R}^3 \to V$ by
$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3.$$
   (a) [5 points] Prove that T is a *linear transformation.*
   (b) [10 points] Prove that if S is linearly independent, then T is one-to-one (injective).
   (c) [10 points] Prove that if $\text{Span}(S) = V$, then T is onto (surjective).

8. Let $A = \begin{bmatrix} -4 & 3 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
   (a) [10 points] Determine the eigenvalues of A.
   (b) [15 points] For a basis for the eigenspace of each eigenvalue.