Fifty[1] Lessons in Basic Topology

MATH455 Course Materials[2][3]

with notes by Jonathan Che ′18

Spring 2017

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[1] Due to a snow storm, the lesson on February 9 was cancelled. So a more proper title would be *Fourty-Nine Lessons in Basic Topology.*

[2] arranged in chronological order.

[3] Solution to homework problems are not included.

Pre-semester Survey[4]

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To know more about your background so that we can try to tailor the course to fit your study goal as much as possible, it would be great if you can answer the following questions by replying to this email.

(1) What is/are your major area(s) of study? And what is/are the area(s) you wish to study but haven't found a chance to do so?

(2) Name one or two (or more) theories/theorems/concepts etc. you learned in your previous academic career (not limited to mathematics) that actually mean(s) something (e.g., being beautiful/elegant/ingenious) to you. Explain why you feel that way, if you can.

(3) What courses have you taken in mathematics? Have you taken a course in group theory? If not, did you come across the definition of a group from somewhere else?

(4) Did you have previous experience with topology? Whether you had or not, what is your own conception of the word/subject topology?

(5) What (topics/methods) do you want to learn the most out of a course in topology, if there is any?

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[4] Inspired by David Foster Wallace's teaching materials. Reference: The David Foster Wallace Reader, 1e, Reader, Little, Brown and Company, 2014.

MATH 455-01 Calendar, Spring 2017

	Monday **11:00 A.M.** **SMUD 207**	**Tuesday**	**Wednesday** **11:00 A.M.** **SMUD 207**	**Thursday** **1:00 P.M.** **MERR 403**	**Friday** **11:00 A.M.** **SMUD 207**
Week 1	Jan 23 **L1** Introduction	Jan 24	Jan 25 **L2** 2.1	Jan 26 **L3** 2.1	Jan 27 **L4** 2.2
Week 2	Jan 30 **Quiz #1** **L5** 2.2	Jan 31	Feb 1 **L6** 3.1	Feb 2 **Hw#1 due** **L7** 3.2	Feb 3 **L8** 3.3
Week 3	Feb 6 **Quiz #2** **L9** 3.3	Feb 7	Feb 8 **L10** 3.4	Feb 9 snowstorm no class	Feb 10 **Hw#2 due** **L11** 3.5
Week 4	Feb 13 **Quiz #3** **L12** 3.5	Feb 14	Feb 15 **L13** 3.6	Feb 16 **Hw#3 due** **L14** 4.1, 4.2	Feb 17 **L15** 4.2
Week 5	Feb 20 **Quiz #4** **L16** 4.2	Feb 21	Feb 22 **L17** 4.3	Feb 23 **Hw#4 due** **L18** 4.3	Feb 24 **L19** L5.1
Week 6	Feb 27 **Quiz #5** **L20** 5.2	Feb 28	Mar 1 **L21** 5.3	Mar 2 **Hw#5 due** **L22** 5.3	**Mar 3** **Exam 1** **2.1 — 4.3**
Week 7	Mar 6 **Quiz #6** **L23** 5.4	Mar 7	Mar 8 **L24** 5.4	Mar 9 **Hw#6 due** **L25** 5.5	Mar 10 **L26** 5.7
Week 8	Mar 13 <—————————	Mar 14 ——————————	Mar 15 Spring Recess	Mar 16 ——————————	Mar 17 —————————>
Week 9	Mar 20 **Quiz #7** **L27** 6.1	Mar 21	Mar 22 **L28** 6.2	Mar 23 **Hw#7 due** **L29** 6.3	Mar 24 **L30** 6.4
Week 10	Mar 27 **Quiz #8** **L31** 6.4	Mar 28	Mar 29 **L32** 7.1	Mar 30 **Hw#8 due** **L33** 7.2	Mar 31 **L34** 7.3
Week 11	Apr 3 **Quiz #9** **L35** 7.4	Apr 4	Apr 5 **L36** 7.5	Apr 6 **Hw#9 due** **L37** 8.1	Apr 7 **L38** 8.2
Week 12	Apr 10 **Quiz #10** **L39** 8.3	Apr 11	Apr 12 **L40** 8.3	Apr 13 **Hw#10 due** **L41** 8.4	**Apr 14** **Exam 2** **5.1 — 7.5**
Week 13	Apr 17 **Quiz #11** **L42** 8.5	Apr 18	Apr 19 **L43** 8.6	Apr 20 **Hw#11 due** **L44** 9.1	Apr 21 **L45** 9.2
Week 14	Apr 24 **Quiz #12** **L46** 9.4	Apr 25	Apr 26 **L47** 10.1	Apr 27 **Hw#12 due** **L48** 10.2	Apr 28 **L49** 10.2
Week 15	May 1 <—————————	May 2 Makeup Days	May 3 —————————>	May 4 <———Reading/Study	May 5 Period——————>

MATH 455-01, Spring 2017: Topology

Class meetings: **MWF** 11:00 – 11:50, Seeley Mudd 207; **Th** 1:00 – 1:50, Merrill 403.
Instructor: Yongheng Zhang
Office: Converse Hall 307
Office Hours: **M** 1:30 – 3:00; **W** 4:00 – 6:00; **Th** 10:30 – 12:00; or by appointment.
Email: yzhang@amherst.com
Text: M. A. Armstrong, *Basic Topology*, Undergraduate Texts in Mathematics, Springer.
Two copies of the textbook are reserved in the science library.

Description: On the first level, **topology is the study of shapes of (topological) spaces**. The most familiar space from single-variable calculus or basic analysis is the real line $\mathbb{R}$ equipped with the standard topology. (*Warning*: there are other topologies on $\mathbb{R}$.) But shapes are not limited to $\mathbb{R}$. Think about the circle S^1. Fourier series are actually defined on it. S^1 is an example of a huge collection of spaces called differentiable manifolds on which you can also do analysis. (Without these intellectual endeavors, general relativity wouldn't have been discovered and thus GPS wouldn't have been as accurate as it has been.) Topology also studies other types of spaces, which are not locally as nice as manifolds (e.g., most of the letters in the English alphabet) as well as more exotic shapes like fractals.

In calculus and analysis, it wouldn't be much fun only to study the real line $\mathbb{R}$ itself. There, functions from $\mathbb{R}$ (or a subset of it) to $\mathbb{R}$ are the main objects of study, where continuity is usually the first property to impose. Extending this idea, on the second level, **topology is the study of (continuous) functions/maps between spaces.** For example, a knot as a space itself is a circle S^1, but even from intuition, there are many different types of knots. In fact, a knot can be seen as a map from S^1 to $\mathbb{R}^3$ (or some equivalence class of it). To take a famous example, we will see in class that any continuous function f from the two-dimensional closed unit disk D^2 in $\mathbb{R}^2$ to itself must have a fixed point. This means that there is at least one $x \in D^2$ such that $f(x) = x$. This is called the two-dimensional Brouwer fixed point theorem. John Nash gave two proofs of his equilibrium theorem, one using a higher dimensional version of the Brouwer fixed theorem and the other using the Kakutani's fixed point theorem, which earned him a Nobel Prize in Economics. (Within the circle of mathematics, he is more famous for his much more difficult theorem on isometric embedding of Riemannian manifolds.)

On the third level, **topology is the study of maps between maps,** which are called **homotopies**. There are just way too many maps between two spaces. But if we do not distinguish two maps whenever there is a homotopy between them, then there are usually just discretely many of them. These lead to computable structures. Fundamental groups and higher dimensional versions of them are well-known but still-far-from-well-studied examples. Homologies are also good algebraic structures. Though it's harder to define homologies, it's easier to compute them.

Then there are maps between maps between maps. This pattern continues *ad infinitum.* But this is area of current research, which is still in its infancy. We stop on the third level.

As physicists who categorize the fundamental particles, chemists who arrange atoms in the periodic table and biologists who put trees in family, genus and species, mathematicians, who share the collector instinct, also classify topological spaces. There are two ways to do it. One do not distinguish between either **homeomorphic** spaces or **homotopy equivalent** spaces. These are second and third level notions, respectively. **Topological** or **homotopy** invariants, e.g., fundamental groups and homologies mentioned above, are used to do the classification once spaces from certain collection have been enumerated.

Topics: We will study topological spaces, continuous maps, compactness, connectedness and path connectedness in their most general form. Then creating new spaces from the old (subspace topology, product topology, quotient topololgy) will be our next step. Afterwards, we will scrutinize homotopy, homotopy type and fundamental group. After the theory of triangulation is developed, the problem of the classification of surfaces will be solved. Then we study homology, which is the foundation for the new area called topological data analysis. Applications are abundant. For example, knots and links can be studied by the tools developed.

Grading: Your grade will be determined by the weighted scores as follows:
Midterm 1 **20%**
Midterm 2 **20%**
Final exam **35%**
Homework **20%**
Quiz **5%**

Attendance: You are expected to attend **every** class, because every lecture is essential to your understanding of topology. If you have to miss a class for medical, religious, or the like reasons, let me know in advance.

Taking Notes: You are expected to take careful notes for this class. One reason is that much of what we will explore in class is not in the textbook. Another reason is that most problems in quizzes, homework and exams are taken either from the homework or from the notes.
But a more important reason is that for a comprehensive course like topology, it is important to follow the narrative and to build your panoramic view of the landscape. If you do not constantly review your notes (I would rather say your journal) and to think about what is happening, it is easy to get lost.

Exams: Midterm 1: **Friday, March 3, in class.**
Midterm 2: **Friday, April 14, in class.**
Final exam: To be announced.
Only pencils and an eraser/ pens are allowed in exams.
Abide by the **Statement of Intellectual Responsibility**.

Homework: Doing homework is the **most important** part of this class. One can only learn mathematics by getting hands dirty. Homework problems will be posted in Moodle for each class day. And problems assigned each week (Monday, Wednesday, Thursday and Friday) will be due the Thursday of the following week. See the calendar for the precise due dates. There are **12** homework sets in total. You must do all the problems from all homework sets in order to excel on the exams.

Start working on the problems **as soon as possible**. Working in groups is **highly recommended**: you can seek help from each other and we usually understand our knowledge better by explaining it to others. However, I suggest you get together only **after** you have spent time thinking about each problem on your own.
You are also very welcome to go to my office hours or send me an email if you have questions.

Your homework solution must be totally **your own work**. That means you must write down the solution **in your own words**, without looking at your group members' work. Copying other's work is considered a violation of the **Statement of Intellectual Responsibility**.

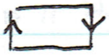
As a courtesy to your grader and for your own benefit of developing neat writing styles, please (1) do the problems in increasing order as listed in Moodle; (2) write in complete mathematical sentences; (3) write legibly (it will be particularly pleasing to everyone if you strive for the standard of calligraphy); (4) write your name on each page and **staple** them in order.

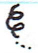
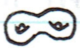
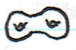
Late Homework: Homework sets are due at the **beginning** of due date classes.
If you expect illness or emergency will prevent you from submitting your homework on time, let me know **before** the due dates so that we can make arrangements without penalty. However, **late homework (not to be turned in at the beginning of due day class) without the above excuses will receive score zero!**

Quizzes: Starting from the second week, there will be a very short quiz at the end of every Monday class. It tests basic concepts introduced the previous week. See the calendar for the dates. Quiz only counts 5% toward your score. Its purpose is to help you keep up with the progression of the course.

1/23 What do we study in topology?

- Level 1: Topological Spaces
  - ex: $\mathbb{R}$, $[0,1]$
  - ex: $S^1 =$  $(x^2+y^2=1)$, $S^2 =$  $(x^2+y^2+z^2=1)$

    "2D": 2 #s describe it.
  - ex: $T^2 =$  $(z^2 = a^2 - (c-\sqrt{x^2+y^2})^2$,  $T^2 \# T^2$
  - ex:  =  arrows = "gluing" in direction given. ,  = 

    Möbius strip: 1 half-twist (180°)
- Level 2: Continuous Functions (Maps) between Spaces
  - ex: $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$
  - ex: $f : \mathbb{R} \to S^1$ by $\theta \mapsto e^{i\theta}$ [covering map:  → ]
  - ex:  $\xrightarrow{\text{Gauss map}}$  by mapping normal vectors of surface to sphere (in S^2)
  - ex:  →  (in $\mathbb{R}^3$)
  - ex:  →  (w/ full twist) [homeomorphic spaces, different embeddings]
- Level 3: "Maps" between Maps - Homotopies ("1-second movies")
  - ex: $f : [0,1] \to$  on D^2 "frame 0"

    $\downarrow t$

    $g : [0,1] \to$  on D^2 "frame 30"

    $\Rightarrow$  (all frames in one picture)

Ex: cut Möbius strip along center circle → 1 loop, 4 half-twists

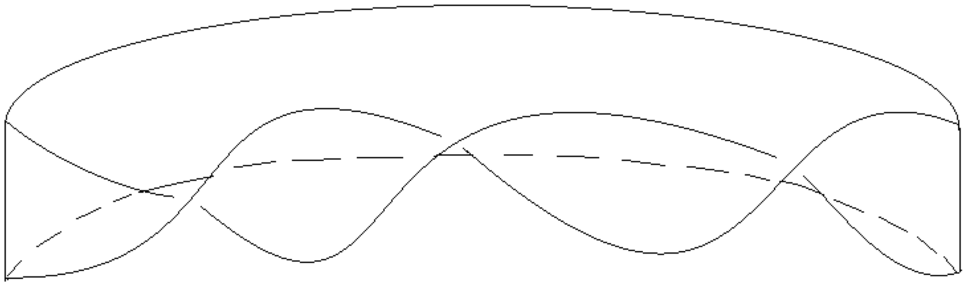
Ex: cut along 1/3 line → 2 linked loops: 1 half-twist, 4 half-twist (& 2x length)

Ex: cut strip w/ 2 half-twists along center → 2 linked loops, 2 half-twists each.

Problem for Lesson 1: Introduction

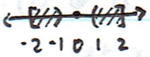
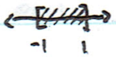
January 23, 2017

1. Construct a strip with three "left-handed" half-twists in the following way. The *Beginning Topologist's Toolbox* you got in class today is definitely of help. If you do not have one, let me know. It's a simple trip to Walmart, Target and Jo-Ann Fabrics and Crafts for me.



This is the image of an embedding of the usual Möbius strip (with "one half-twist") into the three dimensional world we live in. Now cut the strip along the central circle. Ignoring the thickness (and thus the twists), what do you get? (Before doing the cutting, try to imagine what you would get.) Google "knot theory" and then read the Wikipedia article with the same title. Is your knot the same as the one you saw in the first two pictures there? Check out the last two pictures in Section 4.1 of this article to confirm your answer. **So what is the precise name of your knot?** (You only need to record your answer to this last question for this problem for the homework you will turn in next Thursday.)

1/25 Motivation for Topology / Topological Spaces

- ex:  $\overset{\text{"1:1"}}{\underset{\text{"onto"}}{\hookrightarrow\!\!\!\rightarrow}}$  by $f(x) = \begin{cases} x+1 & -2 \leq x < 1 \\ 0 & x = 0 \\ x-1 & 1 < x \leq 2 \end{cases}$

  ↳ though we start with 3 pieces and end up with 1 piece, these sets are still considered the same.

- ex: $f: [0, 2\pi) \hookrightarrow\!\!\!\rightarrow S^1$
  $\theta \longmapsto e^{i\theta}$

  ↳ a circle has a hole in the middle - how is it the same as a line segment?

- ex: $f: \mathbb{R} \hookrightarrow\!\!\!\rightarrow \mathbb{R}^2$ by Bernstein-Schroder construction

  ↳ clearly pathological: 1D space shouldn't be the same as 2D space.

$\Longrightarrow$ in all of these cases, we are looking at "geometric" objects on the level of sets, which are discrete w/ no structure.

Def: A (topological) space is a set X together with a collection of subsets $\mathcal{T}$ of X s.t.:

"T1" ① $\phi, X \in \mathcal{T}$

"T2" ② For any $O_1, O_2 \in \mathcal{T}$, $O_1 \cap O_2 \in \mathcal{T}$

"T3" ③ For arbitrarily many $O_i \in \mathcal{T}$, $\bigcup O_i \in \mathcal{T}$

- $\mathcal{T}$ is called a topology on X
- each element in $\mathcal{T}$ is called an open set
- $(X, \mathcal{T})$ is called a topological space [or just X if $\mathcal{T}$ is clear]

(recall condition ②)

Thm: ② $\Longleftrightarrow$ For finitely many $O_1, \ldots, O_n \in \mathcal{T}$, $O_1 \cap \ldots \cap O_n \in \mathcal{T}$

Pf: ($\Leftarrow$) $n = 2$ ✓

($\Rightarrow$) $O_1 \cap O_2$ is open

Then $(O_1 \cap O_2) \cap O_3$ is open

$\ldots (O_1 \cap \ldots \cap O_{n-1}) \cap O_n$ is open □

Ex: (Standard Topology on $\mathbb{R}$) O is open if $\forall x \in O, \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq O$

Pf: ① $\emptyset$ is vacuously open

Let $x \in \mathbb{R}$.

Then $(x-1, x+1) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is open.

② Let O_1, O_2 be open.

Let $x \in O_1, x \in O_2$ (i.e. $x \in O_1 \cap O_2$)

Then $\exists \varepsilon_1, \varepsilon_2 > 0$ s.t. $V_{\varepsilon_1}(x) \subseteq O_1$, $V_{\varepsilon_2}(x) \subseteq O_2$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

So $V_{\varepsilon}(x) \subseteq V_{\varepsilon_1}(x)$ and $V_{\varepsilon}(x) \subseteq V_{\varepsilon_2}(x)$

So $V_{\varepsilon}(x) \subseteq V_{\varepsilon_1}(x) \cap V_{\varepsilon_2}(x) \longrightarrow O_1 \cap O_2$ is open

③ Let each O_i be open.

Let $x \in \bigcup O_i$

Then $x \in O_j$ for some $j \in$ index set.

Since O_j open, $\exists \varepsilon > 0$ s.t. $V_{\varepsilon}(x) \subseteq O_j \subseteq \bigcup O_i \longrightarrow \bigcup O_i$ is open □

Note: (a,b) is open

Note: An arbitrary open set in $\mathbb{R}$ is a union of open intervals.

"finite component"

Ex: ($\mathbb{R}_{fc}$) O is open if either O is $\emptyset$ or $\mathbb{R} \setminus O$ is finite

Pf: ① $\emptyset$ is open by definition

$\mathbb{R} \setminus \mathbb{R} = \emptyset$ which is finite $\longrightarrow \mathbb{R}$ is open

② Let O_1, O_2 be open

If one is $\emptyset$, then $O_1 \cap O_2 = \emptyset \longrightarrow O_1 \cap O_2$ is open

If neither is $\emptyset$, then $\mathbb{R} \setminus O_1$ and $\mathbb{R} \setminus O_2$ are open.

Then $\mathbb{R} \setminus (O_1 \cap O_2) = (\mathbb{R} \setminus O_1) \cup (\mathbb{R} \setminus O_2)$ which is finite $\longrightarrow O_1 \cap O_2$ is open

③ Let each O_i be open.

If all O_i are $\emptyset$, then $\bigcup O_i = \emptyset$ is open

Suppose $O_j \neq \emptyset$. Then $\mathbb{R} \setminus O_j$ is finite

Then $\mathbb{R} \setminus (\bigcup O_i) = \bigcap (\mathbb{R} \setminus O_i) \subseteq \mathbb{R} \setminus O_j \longrightarrow \bigcup O_i$ is open □

Problems for Lesson 2: Topological Spaces

January 25, 2017

The purpose of homework is to enhance your understanding of the notions, theorems and theories introduced in class and to enable you to apply them to new situations. There are several ways to do that. For example, in today's homework, you will explore an alternative way to define a topological space, check why certain axiom has to be in that way and to apply your understanding to a simple example and a slightly more complicated example.

1. Given a topological space X (The collection $\mathcal{T}$ doesn't have to be written out explicitly. But since it says *topological space*, a collection of open sets is assumed to exist.), a subset C is said to be **closed** if its complement $X \backslash C$ is open. Prove that if X is a topological space, then
   (1) $\emptyset$, X are both closed;
   (2) if C_1 and C_2 are closed, then $C_1 \cup C_2$ is also closed;
   (3) if C_i, $i \in I$ are closed, then $\cap_{i \in I} C_i$ is also closed.

   Hint: the De Morgan's law we used in the $\mathbb{R}_{fc}$ example in class is most of what you need.

   In fact, the above properties of closed sets also imply the defining properties of open sets. (You don't need to prove this.) So a topological space can be equivalently defined via closed sets.

2. In the definition of a topological space, we only require that the intersection of **two** (equivalently, **finitely many**) open sets is open. Give an example of **infinitely many** open sets in $\mathbb{R}$ with the standard topology such that their intersection is not open. (If you forget how to do/have never done this little exercise in real analysis, then you can easily find its answer by Googling.)

3. Is the following a topological space? Prove your claim.
$$X = \{1, 2, 3, 4\}, \mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$$

4. Let X be $\mathbb{R}^2$. Given $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in X, recall that their Euclidean distance d is given by
$$d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$
   The open ball centered at a with radius ϵ is denoted by $B_\epsilon(a)$, which is defined as $\{x \in \mathbb{R}^2 | d(x, a) < \epsilon\}$. A subset O of X is called open if for any $x \in O$, there is $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq O$. Prove that
   (1) X with the open sets described above indeed is a topological space;
   (2) each open ball is open;
   (3) each open set of X is a union of open balls.

   Hint: Pictures help. We proved analogous results in class for $\mathbb{R}$ with the standard topology.

L3

1/26 Motivation for bases for topologies

∘ recall: for $\mathbb{R}$ w/ standard topology, any open set $O = \bigcup (\text{open intervals})$

↳ we can think of those open intervals as a basis

Def: Given a set X, a base/basis for a topology on X is a collection $\mathcal{B}$ of subsets of X s.t.:

"B1" ① $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$

"B2" ② $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq B_1 \cap B_2$

- note: subsets $B \in \mathcal{B}$ are called basis elements

Def: Let $\mathcal{B}$ be a basis for X. Then the topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as follows: O is open if $O = \bigcup_{i \in I} B_i$ for $B_i \in \mathcal{B}, i \in I$.

Pf: (T1) $\emptyset \in \mathcal{T}$ [either modify definition, or take union of no elements]

$\forall x \in X, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x$.

Then $\bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B_x \subseteq X$, and $X \subseteq \bigcup_{x \in X} \{x\}$

So $X = \bigcup_{x \in X} B_x \rightarrow X$ is open.

(T2) Let O_1, O_2 be open.

So $O_1 = \bigcup_{i \in I} B_i$, $O_2 = \bigcup_{j \in J} B_j$, each $B_i \in \mathcal{B}, B_j \in \mathcal{B}$

Then $O_1 \cap O_2 = \bigcup B_i \cap \bigcup B_j = \bigcup_{\substack{i \in I, \\ j \in J}} (B_i \cap B_j)$

$\forall x \in B_1 \cap B_2, \exists B_x$ s.t. $x \in B_x \subseteq B_i \cap B_j$ [B2]

Then $B_i \cap B_j = \bigcup_{x \in B_i \cap B_j} B_x \rightarrow O_1 \cap O_2$ is open

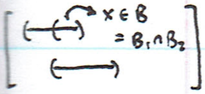
(T3) Let each O_i be open, $i \in I$.

So for each $i \in I$, $O_i = \bigcup_{i_j \in I_i} B_{i_j}$, $i_j \in I_i$

So $\bigcup_{i \in I} O_i = \bigcup_{i \in I} \bigcup_{i_j \in I_i} B_{i_j} \rightarrow \bigcup O_i$ is open □

Ex: For $\mathbb{R}$, $\mathcal{B} = \{(a,b) \mid a<b\}$ is a basis

(B1) $\forall x \in \mathbb{R},\ x \in (x-1, x+1)$



(B2) For any two open intervals, their intersection is again an open interval. This open interval contains x and is the intersection

[basis → top by local ver.]

Thm: Let $\mathcal{T}$ be generated by $\mathcal{B}$. Then $O \in \mathcal{T} \iff \forall x \in O, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq O$.

Pf: ($\Rightarrow$) Let $O = \bigcup_{i \in I} B_i$. Let $x \in O$

So $x \in B_j$ for some $j \in I$.

So $x \in B_j \subseteq \bigcup B_i = O$.

($\Leftarrow$) For any $x \in O$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq O$.

So $O = \bigcup_{x \in O} B_x$ □

Note: $\mathcal{B} = \{(a,b) \mid a<b\}$ is a basis for the standard topology on $\mathbb{R}$.

Note: Basis in topology ≠ Basis in linear algebra.

∘ expression of open set in terms of basis elements is not unique.

Note: Each basis element is itself an open set [$B = B \cup \emptyset$ is open]

Thm: Let $(X, \mathcal{T})$ be a space. Let $\mathcal{C}$ be a subcollection of $\mathcal{T}$. If $\overset{\text{For each}}{\forall} O \in \mathcal{T}, x \in O$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subseteq O$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Problems for Lesson 3: Bases for Topologies

January 26, 2017

(1) (Notes review question) Where was the second defining property of basis used in proving that the topology it generates indeed is a topology?

(2) Let X be the set $\mathbb{R}$. Let each element of $\mathcal{B}$ be an interval of the form $[a, b)$ where $a < b$. Prove that
  (1) $\mathcal{B}$ is a basis (the topology it generates is called the **lower limit topology** of $\mathbb{R}$ and in this case the space is written $\mathbb{R}_l$);
  (2) any open set in $\mathbb{R}$ with the standard topology is an open set in $\mathbb{R}_l$ but not vice versa. In this case, people say that the topology of $\mathbb{R}_l$ is *strictly finer* than the standard topology of $\mathbb{R}$.

(3) Prove the last theorem stated in class: Let $(X, \mathcal{T})$ be a space. Let $\mathcal{C}$ be a subcollection of $\mathcal{T}$. (So elements of $\mathcal{C}$ are open sets.) If for each $O \in \mathcal{T}$, and each $x \in O$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq O$, then $\mathcal{C}$ is a basis for $\mathcal{T}$, which means
  (1) $\mathcal{C}$ is a basis, and
  (2) the topology $\mathcal{C}$ generates coincides with $\mathcal{T}$. (*Hint:* Let the topology $\mathcal{C}$ generates be $\mathcal{T}'$. Show that $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{T}'$.)

(4) Let $X = \mathbb{R}^2$ be the topological space in the last homework problem from yesterday. Let the elements of $\mathcal{C}$ be open rectangles whose edges are parallel to the coordinate axes: $\{(x, y) \in \mathbb{R}^2 | a < x < b, c < y < d\}$. Use the previous theorem to show that $\mathcal{C}$ is a basis generating the same topology.

1/27 Def: Let X, Y be spaces. A function $f: X \to Y$ is continuous if for any open set O in Y, $f^{-1}(O)$ is open in X

- note: we can write $f^{-1}(O)$ as $f^{-1}O$
- note: continuous functions are also called maps (or continuous maps)

Ex: (identity map) Let $id: X \to X$, $id(x) = x$

∘ case 1: If the two X have the same topology, then id is continuous

Pf: Let O be open in X

Then $id^{-1}(O) = O$ is open □

∘ case 2: The two X may have different topologies.

- ex: $id: \mathbb{R} \to \mathbb{R}_\ell$ (lower limit topology: generated by basis $\{[a,b)\}$) is not continuous.

Pf: Consider $[0,1)$, which is open in $\mathbb{R}_\ell$

$id^{-1}[0,1) = [0,1)$ is in $\mathbb{R}$

But $[0,1)$ is not open in $\mathbb{R}$ with the standard topology

For $0 \in [0,1)$, we cannot find $0 \in (-\varepsilon, \varepsilon)$ and $(-\varepsilon, \varepsilon) \subseteq [0,1)$

Ex: (constant functions) Let $f: X \to Y$ be a constant function. Then f is continuous.

Pf: Let O be open in Y. Consider $f^{-1}(O)$

Case 1: If $y_0 \notin O$, then $f^{-1}O = \emptyset$ is open

Case 2: If $y_0 \in O$, then $f^{-1}O = X$ is open □

Ex: (discrete topology) Given set X and space Y, what kind of topology can you put on X s.t. any $f: X \to Y$ is continuous? $X, \mathcal{T} = P(X) = \{\text{subsets of } X\}$

Pf: Let $f X \to Y$ be a function, let O be open in Y.

Then $f^{-1}O \subseteq X$, so $f^{-1}O$ is in $P(X)$, i.e. open

∘ note: (trivial topology) $X, \mathcal{T} = \{\emptyset, X\}$

"basic open set"

Prop: $f: X \to Y$ is continuous $\iff$ $\forall$ basis elements B in Y, $f^{-1}B$ is open in X.

Pf: ($\Rightarrow$) Let B be a basic open set in Y, i.e. B is open.

So $f^{-1}B$ is open

($\Leftarrow$) Let O be open in Y.

Then $O = \bigcup_{i \in I} B_i$ for some basic open sets B_i

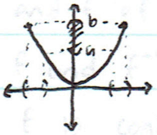
So $f^{-1}(\bigcup B_i) = \bigcup f^{-1}B$, which is union of open sets, i.e. open. $\square$

Ex: $f: \mathbb{R} \to \mathbb{R}$ by $x \mapsto 2x+1$ is continuous

Pf: Let (a,b) be any basic open set in $\mathbb{R}$

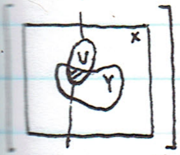
Then $f^{-1}(a,b) = (\frac{a-1}{2}, \frac{b-1}{2})$ is open $\square$

Ex: $f: \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ is continuous



$\to f^{-1}(a,b) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$ is open

U is open in X



U∩Y is open in Y

standard topology in $\mathbb{R}$

Def: Let X be a space. Let $Y \subseteq X$. The subspace topology on Y is defined as follows: O is open in Y if $O = U \cap Y$, where U is a open set in X

- note: we say X induces the subspace topology on Y
- note: we say Y inherits the subspace topology from X

Ex: $O = [0,1)$ is open in $[0, 2\pi)$

Pf: $[0,1) = (-1,1) \cap [0, 2\pi)$ where $(-1,1)$ is open in $\mathbb{R}$

Ex: ($\mathbb{Z}$) The subspace topology on $\mathbb{Z}$ from $\mathbb{R}$ is the same as the discrete topology on $\mathbb{Z}$

Problems for Lesson 4: Continuous Functions

January 27, 2017

(1) Prove that $f : X \to Y$ is continuous if and only if for any closed set C in Y, $f^{-1}(C)$ is closed.
Hint: The set-theoretic fact $f^{-1}(Y \backslash O) = X \backslash f^{-1}(O)$ is useful here.

(2) Recall that in calculus and analysis, $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for any given $\epsilon > 0$, there is $\delta > 0$ such that whenever $x \in \mathbb{R}$ satisfies $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. Then $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if f is continuous at every $x_0 \in \mathbb{R}$.

Prove that for $f : \mathbb{R} \to \mathbb{R}$, it is continuous in the sense of calculus and analysis if and only if it is continuous in the sense of topology. But you saw how easy the definition of continuity is in topology.

Hint: the calculus definition of $f : \mathbb{R} \to \mathbb{R}$ being continuous at x_0 can be slightly reformulated as follows: given any $\epsilon > 0$, there is $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$, $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Furthermore, the characterization of continuity using basic open sets is slightly easier than the definition of continuity using open sets.

You can find the solutions to the above problems (probably for all standard homework problems in topology) within a split second using Google. Do that only when you are stuck but have worked on each problem at least for half an hour. Going through the notes again and reading the textbook (in the future) might be a better first way to look for help. In graduate school, it's not uncommon to work on a homework problem for three months.

MATH 455 Quiz #1 Name:____________________

1. (2 points) Complete the definition: $(X, \mathcal{T})$ is a topological space if

2. (2 points) Complete the definition: $\mathcal{B}$ is a basis for X if

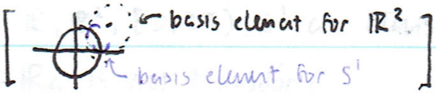
3. (2 points) State the **two** equivalent ways of constructing/generating a topology $\mathcal{T}$ from a basis $\mathcal{B}$.

4. (2 points) Complete the definition: $f : X \to Y$ is continuous if

5. (2 points) This is a survey question. Circle your choice. *How do you feel about the amount of homework assigned?* The homework is
   - way too much.
   - manageable but I would still prefer less.
   - just the right amount.
   - too little and I want more.

1/30 Motivation for homeomorphisms

- ex: $f:[0,2\pi) \hookrightarrow\!\!\!\rightarrow \oplus$ by $\theta \mapsto e^{i\theta}$: is f continuous?
  - ↳ recall that $\{B_\varepsilon(x)\}$ is a basis for $\mathbb{R}^2$.
    - then you can show $\{B_\varepsilon(x) \cap S^1\}$ is a basis for S^1



- Case 1: $f:[0,2\pi) \longrightarrow \oplus$: $f^{-1}(B) = (\frac{\pi}{4}, \frac{3\pi}{4})$ is open
- Case 2: $f:[0,2\pi) \longrightarrow \oplus$: $f^{-1}(B) = [0,\frac{\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$ is open

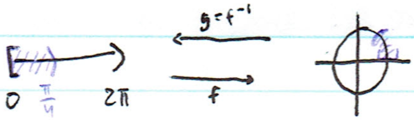
recall: $[0,\frac{\pi}{4})$ open in $[0,2\pi)$ but not in $\mathbb{R}$

↳ By proposition, f is continuous, but f is not a homeomorphism

Def: Let $f: X \to Y$ be a bijection (i.e. $f^{-1}: Y \to X$ exists s.t. $f \circ f^{-1} = id_Y$ [1:1] and $f^{-1} \circ f = id_X$ [onto]) Then f is a homeomorphism if both f and f^{-1} are continuous

- so f is continuous and f has a continuous inverse.
- "Level 1": ∃ bijection b/w points in sets X and Y.
- "Level 2": ∃ bijection b/w the open sets in X and Y.
- note: isomorphism in Group theory ~ homeomorphism in Topology ("same thing")

Ex: the motivating example is not a homeomorphism



: g is not continuous b/c $g^{-1}[0,\frac{\pi}{4})$ is not open.

- 1 is in ⌒, but there is no open curved interval containing 1 in ⌒

think: complex plane.

Def: X and Y are homeomorphic if there is a homeomorphism b/w X and Y.

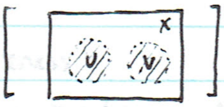
- written as $X \cong Y$ or $X \simeq Y$
- note: X and Y are not homeomorphic iff any function $f: X \to Y$ is not a homeomorphism (must check all candidates)
  - e.g.: motivating example shows f is not a homeomorphism, but does not show that $[0,2\pi)$ and S^1 are not homeomorphic.

Def: A space X is called Hausdorff if $\forall x \neq y \in X$, $\exists$ open sets $U, V \in X$ s.t.:

① $x \in U$

② $y \in V$

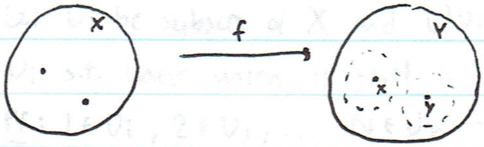
③ $U \cap V = \emptyset$



- ex: $\mathbb{R}, \mathbb{R}^2, [0, 2\pi), S^1$ are Hausdorff
- ex: $\mathbb{R}_{fc}$ is not Hausdorff

Thm: If $f: X \to Y$ is a homeomorphism and X is Hausdorff, then Y is Hausdorff.

Pf:



$\mathbb{R} \not\cong \mathbb{R}_{fc}$

↳ due any homeomorphism

Problems for Lesson 5: Homeomorphism

January 30, 2017

Problem (2) will be graded.

Just for this time, 4 points will be assigned for completion of problems from L5. How Hw#1, Hw#2 and Hw#n for $3 \leq n \leq 12$ will be graded is explained in the email titled "a few things about MATH 455".

(1) Let Y be a subspace of X. Recall that this means O is an open set in Y if and only if there is an open set U in X such that $O = U \cap Y$. Prove that this indeed gives a topology for Y.

(2) Let Y be a subspace of X and $\mathcal{B}$ a basis for the space X. Prove that $\{B \cap Y | B \in \mathcal{B}\}$ is a basis generating the subspace topology of Y.
Hint: Use Problem (3) from Lesson 3 (last Thursday).

(3) Who is this mathematician? What is the title of his Ph.D. thesis in its original language?

(4) Prove (again) the last theorem we stated in class: If $f : X \to Y$ is a homeomorphism and X is Hausdorff, then Y is also Hausdorff.

(5) Prove that being homeomorphic is an equivalence relation. (Thus, we can form equivalence classes called **topological types** of topological spaces.) This means
  (a) $id : X \to X$ is a homeomorphism;
  (b) if $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$;
  (c) if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $g \circ f : X \to Z$.

2/1 Introduction to Compactness

Compactness "=" Finiteness

- For a finite set, wlog $X = \{1, \dots, N\}$ for some N. Then:
  ① Any function $f: X \to \mathbb{R}$ attains max/min
  ② Any sequence $\{x_n\}_{n=1}^{\infty}$ in X has a convergent subsequence, since there must be a constant subsequence that converges.
  ③ X has a discrete topology → X is closed and bounded
  ④ Let U_i be subsets of X and $\bigcup U_i = X$. Then $\exists$ finite subcollection of U_i s.t. their union is still X
  Pf: $1 \in U_i$, $2 \in U_j$, ..., $N \in U_k$ → $\bigcup_{j=1}^{N} U_{i_j} = X$

Consider $[0,1]$ (a subspace of $\mathbb{R}$)
① Any continuous $f: [0,1] \to \mathbb{R}$ attains max/min [extreme value thm]
② Any $\{x_n\}$ in $[0,1]$ has convergent subsequence [Bolzano-Weierstrass]
③ $[0,1]$ is closed and bounded in $\mathbb{R}$. [Heine-Borel]
④ If $[0,1] \subseteq \bigcup O_i$ (open sets in $\mathbb{R}$), then we're always able to find finitely many such O_is s.t. their union also contains $[0,1]$.

Def: Let X be a space. Let $A \subseteq X$. Let $\mathcal{O} \subseteq \mathcal{T}$ (some collection of open sets). $\mathcal{O}$ is an open cover of A if $A \subseteq \bigcup_{O \in \mathcal{O}} O$. If $\mathcal{O}' \subseteq \mathcal{O}$ and $A \subseteq \bigcup_{O \in \mathcal{O}'} O$, then $\mathcal{O}'$ is a subcover of $\mathcal{O}$.

Def: The space X is compact if every open cover has a finite subcover

- e.g. any finite set with discrete topology is compact
- e.g. $[0,1]$ is compact.
- e.g. $\mathbb{R}$ is not compact.

Pf: $\{(n, n+2) \mid n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$.
Removing any open interval makes $\mathbb{R}$ not fully covered.

Def: Let X be a space. Let $A \subseteq X$. A is compact in X if A is compact in the subspace topology of X.

Lemma: A is compact in X <=> for any open cover of A by open sets in X, it has a finite subcover.

- e.g. $\{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ is compact in $\mathbb{R}$.

  Pf: Any basic open set of $\mathbb{R}$ that contains $\{0\}$ also contains almost all of the elements in $\{\frac{1}{n}\}$.

  LTR constructing subcover.

Note: Compactness is a topological invariant

- same as Hausdorffness.

Thm: Let $f: X \to Y$ be continuous. If X is compact, then $f(X)$ is compact in Y.

Pf: Let $O_i, i \in I$ be an open cover of $f(X)$ in Y.

Then $f^{-1}O_i$ are open [f continuous]

So $f^{-1}O_i, i \in I$ form an open cover of X. $[X = \bigcup_{i \in I} f^{-1}O_i]$

Since X is compact, say $f^{-1}O_{i_1}, \dots, f^{-1}O_{i_N}$ cover X.

Then $O_{i_1}, \dots, O_{i_N}$ covers $f(X)$ $\square$

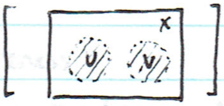
Note: we now see that $(0,1)$ and $[0,1]$ cannot be homomorphic.

Def: A space X is called <u>Hausdorff</u> if $\forall x \neq y \in X$, $\exists$ open sets $U, V \in X$ s.t.:

① $x \in U$

② $y \in V$

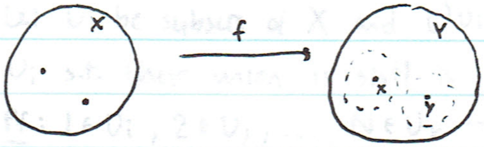
③ $U \cap V = \emptyset$



- ex: $\mathbb{R}, \mathbb{R}^2, [0, 2\pi), S^1$ are Hausdorff
- ex: $\mathbb{R}_{fc}$ is not Hausdorff

Thm: If $f: X \to Y$ is a homeomorphism and X is Hausdorff, then Y is Hausdorff.

<u>Pf</u>:



$\mathbb{R} \not\cong \mathbb{R}_{fc}$
↳ due <u>any</u> homeomorphism

Problems for Lesson 6: Introduction to Compactness

February 1, 2017

Problem (3) will be graded.

This is NOT a long homework set, even though it exceeds one page. It looks verbose only because it tries to make things easier with elaboration.

(1) We can also use closed sets to characterize compactness. Prove that X is compact if and only if given a collection of closed sets C_i, $i \in I$ in X, if for any finite sub-collection, the intersection of its elements is nonempty, then $\cap_{i\in I} C_i$ is also nonempty.

Hint: First use the definitions of closed set and the De Morgan law to prove that X is compact if and only if given any collection of closed sets C_i, $i \in I$ in X, *if $\cap_{i\in I} C_i$ is empty, then there is a finite sub-collection such that the intersection of its elements is also empty*. Then apply contrapositive to the *blue italic clause*.

(2) Let A be a subset of the topological space X. Recall that A is said to be compact in X if A is compact in the subspace topology of X, which means for any open cover of A by open sets in the subspace topology of A, it has a finite subcover.

Prove that A is compact in X if and only if for any open cover of A by open sets in X, it has a finite subcover.

Hint: Recall O is open in A means there is U open in X such that $O = U \cap A$.

(3) **Definition.** A space X is called ***locally compact*** if for any $x \in X$, there is an open set U in X and a compact subset K in X such that $x \in U \subseteq K$.

(a) Prove that any compact space is locally compact.

(b) Prove that $\mathbb{R}$ (with the standard topology) is not compact but it is locally compact.

(4) This is mainly a reading problem.

Definition. Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbb{R}$ satisfying three properties:
M1 $d(x, y) \geq 0$ for any $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
M2 $d(x, y) = d(y, x)$ for any $x, y \in X$.
M3 $d(x, y) + d(y, z) \geq d(x, z)$ for any $x, y, z \in X$.

The metric d is also called a **distance function**. It abstracts and generalizes the usual notion of Euclidean distance. M1 means any value of distance should be nonnegative and if two points occupy the same location, then their distance should be zero and vice versa. M2 means the distance from x to y should be the same as the distance from y to x. This is sometimes called the symmetric property. M3 means distance satisfies the triangle inequality. Examples abound. $\mathbb{R}^n$ for any $n \in \mathbb{N}$ has the Euclidean metric. You can also put other metrics on them. For example, google **Taxicab metric**. It's used in compressed sensing. This metric was invented by German mathematician Hermann Minkowski, who was Albert Einstein's math teacher at nowadays ETH Zurich. He recast Einstein's theory of special relativity in the differential geometric language of space-time, which is still used today.

As another example, let $C[0,1]$ be the set of all the continuous functions from $[0,1]$ to $\mathbb{R}$ (or $\mathbb{C}$). We can define $d(f,g)$ as $\int_0^1 |f(x) - g(x)| dx$. Using the properties of integral, one can show that d is indeed a metric (meaning satisfying M1,M2 and M3).

What is metric good for? For us, it can be used to define a topology.

Here is how it is done. Given (X,d), let $B_\epsilon(x)$ be defined by $\{y \in X | d(x,y) < \epsilon\}$. We call it the open ball centered at x with radius some positive number ϵ. Let $\mathcal{B}$ be the collection of all such open balls. You can check that $\mathcal{B}$ is a basis. Notice that the open balls are abstract: they don't necessarily look like open balls in $\mathbb{R}^2$. You need to use $M1, 2, 3$ to check this.

Definition. The metric topology for (X,d) is the one generated by the above basis. With this topology, X is called a metric space.

Recall how they were defined: $\mathbb{R}$ is actually a metric space for the distance function $|x-y|$ and $\mathbb{R}^2$ a metric space for the Euclidean distance. $C[0,1]$ under that "integral metric" also becomes a topological space. This is our first nontrivial **function space** you see in this course. Its elements are functions. We say it's nontrivial because fixing a singleton $\{a\}$, given any set X and any $x \in X$, x can be viewed as a function from $\{a\}$ to $\{x\}$.

An immediate property of metric space is that if X is a metric space, then it is **Hausdorff**. You can show that given any $x \neq y \in X$, the two basic open sets $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$, where ϵ is the distance between x and y, are disjoint using the properties of the metric function.

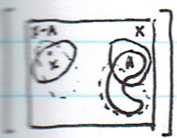
Lastly, using the above property, we now know that not every topological space admits a metric. Think about $\mathbb{R}_{fc}$. You checked that it's not Hausdorff. So it is not a metric space. This means there is no metric d on the set $\mathbb{R}$ inducing the finite complement topology. People sometimes say $\mathbb{R}_{fc}$ is not **metrizible**.

THE END.

2/2 Hausdorff / Compact / Closed / Homeomorphism

not closed & bounded: X may not be metrizable.

1 Thm: Let X be a Hausdorff space. Let A be compact in X. Then A is closed in X.



Pf: [To show A is closed, show $X \setminus A$ is open
It suffices to show that any point in $X \setminus A$ is contained in an open set which is contained in $X \setminus A$.]

Let $x \in X \setminus A$.

Let $a \in A$.

So $x \neq a$. Because X Hausdorff, $\exists$ open sets $U_a(\ni x)$, $V_a(\ni a)$ s.t. $U_a \cap V_a = \emptyset$

The collection $\{V_a \mid a \in A\}$ is an open cover of A.

Since A compact, $\exists$ finite subcover of $\{V_a \mid a \in A\}$, $\{V_{a_1}, \dots, V_{a_N}\}$.

Let $V = \bigcup_{i=1}^{N} V_{a_i}$

Let $U = \bigcup_{i=1}^{N} U_{a_i}$ which is open. [finite intersection of open sets]

Since $x \in U_{a_i}$ for $i = 1..N$, $x \in U$.

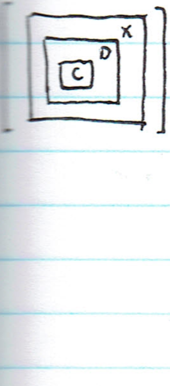
So $U \cap V = (\bigcap_{i=1}^{N} U_{a_i}) \cap (\bigcup_{j=1}^{N} V_{a_j}) = \bigcup_{j=1}^{N} [(\bigcap_{i=1}^{N} U_{a_i}) \cap V_{a_j}] = \bigcup\bigcap[U_{a_i} \cap V_{a_j}]$

$\subseteq \bigcup_{j=1}^{N} U_{a_j} \cap V_{a_j} = \emptyset$ $[U_a \cap V_a = \emptyset]$

Since $A \subseteq V$, and $U \cap V = \emptyset$, then $U \cap A = \emptyset$

So $U \subseteq X \setminus A$.

2 Thm: Let D be compact in space X. Let C be closed in X and $C \subseteq D$. Then C is compact.



Pf: Let Θ be an open cover of C by open sets in X [2/1 Lemma]

Notice that $X \setminus C$ is open [C closed]

Let $\Theta' = \Theta \cup \{X \setminus C\}$, an open cover for D. (and X)

Since D compact, $\exists$ a finite subcover Θ'' of Θ'

Let $\Theta''' = \Theta'' \setminus \{X \setminus C\}$

Then Θ''' is a finite subcover of Θ of C.

Recall: $f:[0,2\pi) \hookrightarrow$  by $\theta \mapsto e^{i\theta}$

∘ note: f is continuous injection ($X=[0,2\pi) \to Y=\mathbb{R}^2$)

∘ note: f is not a homeomorphism onto its image [f^{-1} not continuous]

→ We want to add conditions to X & Y s.t. continuous injections are homeomorphisms

3 Thm: Let $f:X \to Y$ be a continuous injection where X is compact and Y is Hausdorff. Then f is a homeomorphism onto its image.

Pf: [need to show $f^{-1}|_{f(X)}$ is continuous]

[note: $g:W \to Z$ continuous $\iff$ for any C closed in Z, $g^{-1}(C)$ closed in W]

Let C be closed in X.

Since X is compact, $C \subseteq X$, C is compact [prv. thm]

Since f is continuous, $(f^{-1}|_{f(X)})^{-1}(C) = f(C)$ is compact in Y [last thm L6]

Since $f(C)$ compact in Hausdorff Y, $f(C)$ is closed in Y [1st thm L7]

So $f^{-1}|_{f(X)}$ is continuous. [prob. 1 L4] □

Problems for Lesson 7: Hausdorff Space, Compact Set, Closed Set and Homeomorphism

February 2, 2017

Problem (4) will be graded.

(1) Let $f : X \to Y$ be a continuous function and A a subspace of X. Prove that the restriction of f to A is also continuous. (A function consists of three parts: the domain, the codomain and the rule of assignment. If one of them changes, the function is not the original function any more. For this problem, the domain is changed to the subspace A.)

(2) Let C_1 and C_2 be compact subsets in the space X. Show that $C_1 \cup C_2$ is also compact in X. (This is equivalent to saying the union of finitely many compact subsets of X is also compact.)

(3) Give an example of infinitely many compact subsets of $\mathbb{R}$ whose union is not compact. (Mine example is $\{[n, n+1] : n \in \mathbb{Z}\}$. I believe you must have other examples in mind.)

(4) Let X be Hausdorff and C_i, $i \in I$ an arbitrary collection of compact subsets of X. Prove that $\cap_{c \in I} C_i$ is also compact.

Hint: Use the first two theorems we proved in class today. To get started, notice that since X is Hausdorff, these C_i are closed in X. So $\cap_{i \in I} C_i$ is also closed in X by Problem 1 from L2. From there you can show that it is also compact.

Remark. The Hausdorff property is essential here. You can't do this problem without it.

2/3 Compact Sets in Metric Spaces

(L6: p4) Def: A metric space is a space defined by a metric $d: X \times X \to \mathbb{R}$ s.t:

"M1" ① $d(x,y) \geq 0$

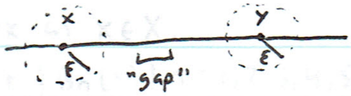
"M2" ② $d(x,y) = d(y,x)$

"M3" ③ $d(x,y) + d(y,z) \geq d(x,z)$

∘ then the basis $B_\varepsilon(x) := \{y \in X \mid d(y,x) < \varepsilon\}$ forms a metric topology

Thm: If X is a metric space, then X is Hausdorff

Pf: (sketch)



Thm: $[a,b]$ is compact in $\mathbb{R}$

Pf: (sketch, full pf in book) $\mathbb{R}$ is complete, i.e. axiom of completeness holds.

So set in $\mathbb{R}$ bounded above has sup, bounded below has inf.

So nested interval property holds

Consider $[a,b]$.

Let O be an open cover of $[a,b]$ by open sets in $\mathbb{R}$

Suppose Θ has no finite subcover.

So if $[a,b]$ is split in half into two intervals, one of those intervals must be covered by infinitely many open sets. So the sequence of nested closed intervals produced in this manner is $I_n = [a_n, b_n]$.

By nested interval property, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Let $x \in \bigcap_{n=1}^{\infty} I_n$

Then $x \in I_1 = [a,b]$.

So $\exists O \in \Theta$ s.t. $x \in O$, i.e. $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq O$

So a $\underbrace{\text{single open set}}_{\text{non-infinite}}$ O covers I_n for $n \geq N \in \mathbb{N}$ ※

Thm: (Heine-Borel) A is compact in $\mathbb{R}$ $\Leftarrow\Rightarrow$ A is closed and bounded in $\mathbb{R}$.

Pf: ($\Leftarrow$) Since A is bounded, $\exists$ $[a,b]$ that contains A.

But A is also closed.

So A is also compact in $\mathbb{R}$. [L7: thm 2]

($\Rightarrow$) We consider a more general theorem

Prop: A is compact in metric space X $\Rightarrow$ A closed and bounded in X.

Pf: X is Hausdorff [X is metric space]

Also, A is compact in X.

So A is closed [L7: thm 1] ✓

Fix an $x \in X$.

Let $\{B_n(x) \mid n = 1,2,3,4,5 \ldots\}$ be an open cover of A.

Since A is compact, $\exists$ finite subcover $B_{n_1}(x) \subseteq \ldots \subseteq B_{n_N}(x)$

So $A \subseteq B_{n_N}(x)$ $\qquad n_1 \le \ldots\ldots \le n_N$

So A is bounded. ✓

(recall analysis def. of compact)

Thm: Let X be a metric space. Let A be compact in X. Let $\{x_n\}$ be a sequence in A. Then $\{x_n\}$ has a subsequence that converges to a point in A.

Pf: (sketch)

$\exists a \in A$ s.t. $\forall \varepsilon > 0$, $B_\varepsilon(a)$ contains infinitely many terms in $\{x_n\}$ [shown below]

↳ choose element of $\{x_n\}$ from each $B_\varepsilon(a)$ in order

$\Rightarrow \{x_n\} \to a$.

Suppose $\forall a \in A$, $\exists \varepsilon_a > 0$ s.t. $B_\varepsilon(a)$ contains finitely many terms in $\{x_n\}$.

$\{B_{\varepsilon_a}(a) \mid a \in A\}$ open cover of A $\Rightarrow$ $B_{\varepsilon_{a_1}}(a_1), \ldots, B_{\varepsilon_{a_N}}(a_N)$ cover A [compact]

Notice $\{x_n\} \subseteq A$, so these $\underbrace{B_{\varepsilon_{a_i}}}_{\text{finite}}$ cover $\underbrace{\{x_n\}}_{\infty \text{ terms}}$. ⨳

Problems for Lesson 8: Compact Sets in Metric Spaces

February 3, 2017

Problem (1) will be graded.

(1) Let $f : X \to \mathbb{R}$ be a continuous function, where X is compact. Prove that f attains its maximum and minimum values. This means that there are $x_1, x_2 \in X$ such that $f(x_1) \geq y$ for all $y \in f(X)$ and $f(x_2) \leq y$ for all $y \in f(X)$.
Hint: See the proof of (3.10) in the textbook.

(2) Who is this mathematician? What's the title of his famous Ph.D. thesis in its original language?

(3) **Definition.** Let A be a subset of the space X. $x \in X$ is called a limit point of A if for every open set O in X which contains x, $(O \backslash \{x\}) \cap A \neq \emptyset$.

Prove that *an infinite subset of a compact space much have a limit point.*
Hint: See (3.8) in the textbook.

(4) Prove the **Lebesgue's Lemma**: Let X be a compact metric space and $\mathcal{O}$ an open cover of X. Then there is a number $\epsilon > 0$ (called a Lebesgue number of $\mathcal{O}$) such that any open ball with radius ϵ in X is contained in some open set from the cover $\mathcal{O}$.
Hint: See (3.11) in the textbook.

MATH 455 Quiz #2

Name:____________________

1. (3 points) Complete the definition: $f : X \to Y$ is a homeomorphism if

2. (4 points) Complete the definition: X is compact if

3. (3 points) Let A be a compact subset of the metric space X. Prove that A is bounded in X, which means A is contained in some open ball in X.

2/6 One-Point Compactification

Recall: Compact spaces have many good properties

- often, we may want to compactify a space since it's easier to study it in its compact version
  - e.g. any (Hausdorff) space ⟶ a compact space (via many methods)

Def: Let X be a Hausdorff space. Let "∞" be a point not in X.
Let $Y = X \cup \{\infty\}$. Define a topology on Y (called the one-point compactification of X) as follows: O is open in Y if either:
① O is an open set in X, or
② $O = Y \setminus C$ (or $\{\infty\} \cup (X \setminus C)$) where C is a compact subset of X

Pf: (T1) Since $\emptyset$ is open in X, $\emptyset$ is open in Y [①]
$\emptyset$ compact in X, so $Y = Y \setminus \emptyset$ is open in Y [②]

(T2) Let O_1, O_2 be open in Y.
Case 1: O_1, O_2 both open in X.
Then $O_1 \cap O_2$ open in $X \rightarrow O_1 \cap O_2$ open in Y [①]
Case 2: $O_1 = Y \setminus C$ (C compact in X), O_2 open in X (wlog)
Then $O_1 \cap O_2 = (Y \setminus C) \cap O_2 = \{\infty\} \cup (X \setminus C) \cap O_2$
Since X Hausdorff, C compact in X, C closed in X [L7, T1]
i.e. $X \setminus C$ open in X, i.e. $(X \setminus C) \cap O_2$ open in X.
So $O_1 \cap O_2 = (X \setminus C) \cap O_2$ open in Y [①]
Case 3: $O_1 = Y \setminus C_1$, $O_2 = Y \setminus C_2$ (C_1, C_2 compact in X)
Then $O_1 \cap O_2 = Y \setminus (C_1 \cup C_2)$
$C_1 \cup C_2$ is compact in X [L7, P2]
So $O_1 \cap O_2$ open in Y [②]

(T3) LTR (homework) □

Note: One-point compactification creates a topological space Y.

Thm: The topology that X inherits from Y is the same as X's original top.

Pf: LTR (homework)

Thm: The one-point compactification Y of X is compact

Pf: LTR (homework)

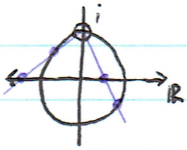
Thm: The one-point compactification Y of X is Hausdorff $\iff$ X is locally compact

Pf: LTR (homework)

Ex: $\underbrace{\mathbb{R} \cup \{\infty\}}_{\text{1-pt comp. of } \mathbb{R}} \cong S^1_z$: $\mathbb{R} \cup \{\infty\}$ compact $\Rightarrow S^1$ compact

point $(0,1)$

- intuition: $\mathbb{R} \cong S^1 \setminus \{i\}$
  - ↳ shown w/ stereographic projection:
  - add $\infty \mapsto i$, $i \mapsto \infty$



$f(t) = \langle \frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \rangle$

: $g(x,y) = \frac{x}{1-y}$

($f : \mathbb{R} \to S^1 \setminus \{i\}$, g opposite)

Problems for Lesson 9: One-Point Compactification

February 6, 2017

Problem (3) will be graded.

(1) Check that the union of an arbitrary collection of open sets in the one-point compactification Y of a Hausdorff space X is open.
Hint: Use Theorems 1 and 2 from L7 and also Problem (4) from L7. Now you also see why we need X be Hausdorff. The following might also be useful. If O is open in X, then there is closed C in X such that $O = X \backslash C$. So $O \cup (Y \backslash K) = (X \backslash C) \cup (Y \backslash K) = (Y \backslash C) \cup (Y \backslash K) = Y \backslash (C \cap K)$.

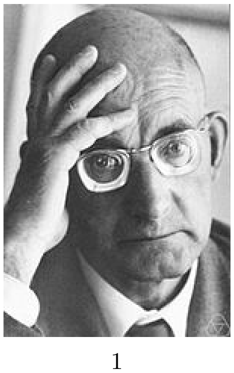
(2) Let X be a Hausdorff space and Y its one-point compactification. Prove that the original topology on X and the subspace topology which X inherits from Y are the same.

(3) Prove that the one-point compactification Y of a Hausdorff space X indeed is compact.
Hint: Any open cover of Y must contain an open set O which contains ∞. Notice that $O = Y \backslash C$ where C is some compact subset of X.

(4) Let X be a Hausdorff space and Y its one-point compactification. Prove that Y is also Hausdorff if and only if X is **locally compact** (introduced in Problem (3) of L6).
Comment: Now you see the usefulness of the definition of local compactness. Not every space is locally compact. For example, $\mathbb{Q}$ is not. The key ingredient in seeing this is the fact that there is an irrational number between any two different real numbers. This is a good example to have in mind.

(5) Show that the one-point compactification of $[0, 1]$ is not homeomorphic to a circle.
Hint: A special singleton is open in this compactified space. This is not the case for a cirlce.

(6) Who is this mathematician? (*Hint:* One-point compactification is also named after him. Without the enlightening of his lifelong mathematician friend and educator Andrey Kolmogorov, the terrain of mathematics wouldn't be as rich as it is now.)



2/8 Product Topology

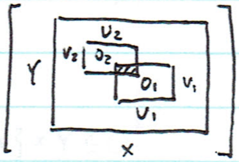
Goal: to create new spaces from old ones

- e.g. by using subspaces (subspace topology)
- e.g. by adding a point (1-pt compactification)
- e.g. by pinching/gluing different points into a single point (quotient topology)
- today: Cartesian product

Def: The Cartesian product of X and Y is $X \times Y := \{\text{ordered } (x,y) \mid x \in X, y \in Y\}$

Q: How do we put a topology on $X \times Y$?

- try: O open if $O = U \times V$ for U open in X, V open in Y



  ⓉⅠ (T1) $\emptyset = \emptyset \times \emptyset$ is open ✓

  $X \times Y$ is open ✓

  (T2) Let $O_1 = U_1 \times V_1$, $O_2 = U_2 \times V_2$ open.

  Then $O_1 \cap O_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ is open.

  (T3) Fails: consider picture for (T2)

  ↳ So it's not a topology, but it is a basis

  (B1) $\forall (x,y) \in X \times Y$, $X \times Y \in \mathcal{B}'$

  (B2) Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}'$

  Then $\forall (x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$,

  $\exists (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}'$ s.t. $\overset{(x,y)}{\cancel{x}} \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$

Def: The product topology on $X \times Y$ is the topology generated by this $\mathcal{B}'$

Thm: Let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for X, Y. Then $\mathcal{B} := \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$

is also a basis for the product topology on $X \times Y$

Pf: LTR (homework)

- e.g. $\mathcal{B} = \{(a,b) \times (c,d)\}$ generates the product topology on $\mathbb{R} \times \mathbb{R} =: \mathbb{R}^2$
  - recall: this is the same as the standard topology on $\mathbb{R}^2$ [L3, P4]

note: it's iff (see book)

Thm: If A, B are compact in X, Y, respectively, then $A \times B$ is compact in $X \times Y$

- e.g. $[a,b] \times [c,d]$ is compact in $\mathbb{R} \times \mathbb{R}$
  - Thm: A subset of $\mathbb{R}^2$ is compact <=> it is closed and bounded

  Pf: (=>) proved in prop in L8

  (<=) Given closed and bounded subset A of $\mathbb{R}^2$

  Then $\underbrace{A}_{\text{closed}} \subseteq \underbrace{[a,b] \times [c,d]}_{\text{compact}}$ is compact [L7, T2]

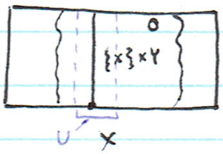
  - e.g. S_1 is compact in $\mathbb{R} \times \mathbb{R}$ (closed & bounded)

Note: everything in this lesson can be generalized to $X_1 \times \ldots \times X_n$ (finite n)

- there is also theory for infinitely many spaces (becomes a function!)

Pf: The 'main ingredient' is the Tube lemma (see homework)

compact Y

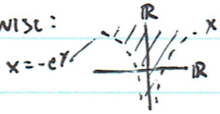


: O open set in $X \times Y$ s.t. $\{x\} \times Y \subseteq O$

↳ then ∃ open set U in X s.t. $x \in U$ and $U \times Y \subseteq O$

- note: Y must be compact

↳ otherwise:



→ clearly, no possible 'tube' (asymptotes)

Problems for Lesson 10: Product Topological Spaces

February 8, 2017

Problem (3) will be graded.

(1) Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topological spaces X and Y, respectively. Prove that

$$\mathcal{B} := \{B_1 \times B_2 \big| B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$$

is a basis generating the product topology on $X \times Y$.

Hint: Use Problem (3) of L3.

(2) Let A and B be subspaces of the topological spaces X and Y, respectively. Prove that the product topology on $A \times B$ is the same as the subspace topology it inherits from the product topology on $X \times Y$.

Hint: $\cup_{i\in I}(U_i \cap A) \times (V_i \cap B) = \cup_{i\in I}(U_i \times V_i) \cap (A \times B)$ is used for proving both directions.

(3) **The Tube Lemma.** Let X and Y be spaces. We also assume Y is compact. Let $x \in X$ and O be an open set in $X \times Y$ such that $\{x\} \times Y \subseteq O$. Prove that there is an open set U in X such that $x \in U$ and $U \times Y \subseteq O$.

Hint: Start as follows. For each $y \in Y$, since $(x, y) \in \{x\} \times Y \subseteq O$ and O is open in $X \times Y$, there are open sets U_y in X and V_y in Y such that $(x, y) \in U_y \times V_y \subseteq O$. The open sets V_y, $y \in Y$ form an open cover of Y. Since Y is compact, you know what to say next. Finally, let U be the intersection of the corresponding finitely many U_{i_j}'s. It is open in X because this is a finite intersection.

(4) **Theorem.** If X and Y are compact spaces, then so is their product $X \times Y$.

Hint: The proof goes in two steps. **Step 1.** Start as follows: Let $\mathcal{O}$ be any open cover of $X \times Y$. Then for any $x \in X$, $\{x\} \times Y$, being homeomorphic to the compact space Y, is also compact. Notice that $\mathcal{O}$ is a cover for $\{x\} \times Y$, so finitely many elements in it cover $\{x\} \times Y$. Let O_x be the union of these finitely many open sets. By the Tube Lemma above, we know there is open set U_x in X such that $\{x\} \times Y \subseteq U_x \times Y \subseteq O_x$. **Step 2.** Now these U_x, $x \in X$ form an open cover of X. Because X is compact, you know what to say next. Since each tube U_x is covered by finitely many open sets from $\mathcal{O}$ and finitely many such tubes cover $X \times Y$, in total, finitely many open sets from $\mathcal{O}$ cover $X \times Y$.

(5) **Theorem.** If A and B are compact subsets of X and Y, respectively, then $A \times B$ is compact in $X \times Y$. *Hint:* This is a direct consequence of (2) and (4).

(6) Prove that the subspace $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ is homeomorphic to the familiar doughnut surface (torus) in $\mathbb{R}^3$.

Hint: Draw pictures.

2/10 Connected Spaces

Def: x is a limit point of A in a space X if for any open set $O \ni x$, the punctured open set $O \setminus \{x\} \cap A \neq \emptyset$

- e.g. $X = \mathbb{R}^2$, $A = \{0\} \cup [1,2]$. Then 0 is not a limit point - it's "detached"
- note: limit points may not be in the set
  - e.g. $X = \mathbb{R}^2$, $A = (0,1]$. Then 0 is a limit point, but $0 \notin A$.

Def: The closure $\bar{A}$ of A is $A \cup \{\text{limit points of } A\}$

Prop: (Alt. def. of closed set) A closed $\Leftrightarrow A = \bar{A}$

Pf: $(\Rightarrow)$ Suppose A is closed.

By definition, $A \subseteq \bar{A}$ (so it suffices to show $\bar{A} \subseteq A$)

Let $x \in \bar{A}$

Suppose $x \notin A$.

So $x \in X \setminus A$, which is open [A closed]

So $\exists$ open set O s.t. $x \in O \subseteq X \setminus A$ (namely, $O = X \setminus A$)

Then $O \cap A = \emptyset$, so clearly $O \setminus \{x\} \cap A = \emptyset$

So x is not a limit point of A

So $x \notin \bar{A}$ $\divideontimes$

$(\Leftarrow)$ LTR (homework) □

Def: X is connected if $\forall A, B \subseteq X$ s.t. $X = A \cup B$, A, B nonempty, $A \cap B = \emptyset$, then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.

- e.g. $X = \mathbb{R}$, $A = (-\infty, 0)$, $B = [0, \infty) \rightarrow \bar{A} \cap B = \{0\} \neq \emptyset$

Thm: $\mathbb{R}$ in the standard topology is connected.

Pf: LTR (in book, thm 3.18)

Thm: A nonempty connected subset of $\mathbb{R}$ is an interval

- note: any interval - e.g. $[a,b]$, $(-\infty, a)$, $[a,b)$, $(-\infty,\infty)$, etc.

Thm: (Equivalent definitions of connectedness) TFAE:

① X is connected

② The only subsets of X that are both open and closed are X and $\emptyset$

Pf: (1=>2) Let A be both open and closed in X.

Let $B = X \setminus A$, also both open and closed. So $X = A \cup B$.

So $\bar{A} = A$, $\bar{B} = B$

Notice $A \cap B = \emptyset$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$

But since X is connected, either $A = \emptyset$ or $B = \emptyset$ [def. connected]

(2=>1) Let A, B be disjoint, nonempty subsets of X s.t. $A \cup B = X$.

[goal: show $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$]

Suppose $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Then $\bar{A} \cap X \setminus A = \emptyset$, so $\bar{A} \subseteq A$

So A is closed [$A \subseteq \bar{A}$, prev. prop.]

Similarly, B is closed, i.e. A is open.

So A is both open and closed, but $\emptyset \subsetneq A \subsetneq X$ ⨳ □

Def: A separation of a space X is a pair C, D of disjoint, nonempty, open subsets of X. whose union is X.

"③" Thm: X is connected <=> X has no separation

Pf: LTR (follows from previous thm)

Thm: Let C, D be a separation of X. Let Y be a connected subset of X. Then either $Y \subseteq C$ or $Y \subseteq D$

Pf: Note: $C \cap Y$, $D \cap Y$ open in Y, $(C \cap Y) \cup (D \cap Y) = Y$, $(C \cap Y) \cap (D \cap Y) = \emptyset$

But since Y is connected, either: $C \cap Y = \emptyset \Rightarrow Y \subseteq D$

or: $D \cap Y = \emptyset \Rightarrow Y \subseteq C$ □

Problems for Lesson 11: Connected Spaces

February 10, 2017

Problem (3) will be graded.

(1) Prove that A is closed in a topological space if and only if $A = \overline{A}$.

(2) Prove that the image of a connected space under a continuous map is connected.

(3) Let A_i, $i \in I$ be connected subspaces of X. Prove that if $\cap_{i\in I} A_i$ is nonempty, then $\cup_{i\in I} A_i$ is also connected.

Hint: For the sake of contradiction, suppose C and D form a separation of $\cup_{i\in I} A_i$. Let $x \in \cap_{i\in I} A_i$. Then $x \in A_i$ for each $i \in I$. Since $x \in \cup_{i\in I} A_i$ and $C \cup D = \cup_{i\in I} A_i$, either $x \in C$ or $x \in D$. Without loss of generality, assume that $x \in C$. Since each A_i is connected, either $A_i \subseteq C$ or $A_i \subseteq D$ by the last theorem we proved in class. Since $x \in A_i$ and $x \in C$, we then must have $A_i \subseteq C$ for all $i \in I$. Then finish the proof.

(4) Prove that S^1 is connected.

Hint: There are many ways to do this. For example, write S^1 as the union of two closed semi-circles. Each is the image of $[0, 1]$ under a continuous map. Then use (2) and (3).

MATH 455 Quiz #3 Name:____________________

1. (4 points) Let X be a Hausdorff space. Define the one-point compactification of X. (You need to define both the set and the topology on it.)

For the next three problems, just write T or F. You don't have to explain.

2. (2 points) True or False? The one-point compactifications of both $[0, 1]$ and $(0, 1)$ are homeomorphic to S^1.

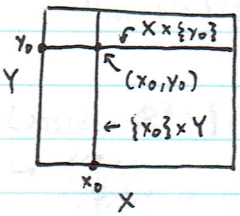
3. (2 points) True or False? If X and Y are compact spaces, then so is $X \times Y$.

4. (2 points) True or False? A space X is connected if and only if there are no pairs of nonempty disjoint open subsets A and B of X whose union is X.

2/13 Connectedness as top. inv.

Thm: If X, Y are connected, then so is $X \times Y$

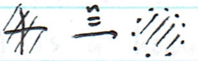
Pf:



Because $\{x_0\} \times Y \cong Y$ and Y connected, $\{x_0\} \times Y$ connected.
Similarly, $X \times \{y_0\}$ is also connected.
Since $X \times \{y_0\} \cap \{x_0\} \times Y = \{(x_0, y_0)\} \neq \emptyset$,
then $X \times \{x_0\} \cup \{x_0\} \times Y$ is connected. [L11, P3]

Note that $X \times Y = \bigcup_{x \in X} \{x\} \times Y \cup X \times \{y_0\}$, each of which is connected.
Clearly, the intersection of these crosses is $X \times \{y_0\}$, i.e. nonempty.
So $X \times Y$ is connected [L11, P3] □

Ex: Since $\mathbb{R}$ is connected, $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ is also connected.
↳ similarly, $\mathbb{R}^3, \mathbb{R}^4, \ldots$ are connected.

Ex: So $\mathbb{R}^n \cong B_1(0)$ by $\vec{x} \mapsto \frac{\vec{x}}{1+|\vec{x}|}, \frac{\vec{y}}{1-|\vec{y}|} \mapsfrom \vec{y}$ [$n=2$: ]
↳ by thm, since $\mathbb{R}^n$ is connected, then $B_1(0)$ is also connected.

Thm: Let A be connected in X. Let $B \subseteq X$ s.t. $A \subseteq B \subseteq \bar{A}$. Then B is connected.
- note: think of B as A w/ some limit points added.

Pf: Suppose B is not connected.
Then $\exists$ a separation C, D of B.
Then $A \subseteq C \cup D \ (= B) \subseteq \bar{A}$
Since A is connected in B, either $A \subseteq C$ or $A \subseteq D$. [L11, last thm]
wlog say $A \subseteq C$.
Then $\bar{A} \subseteq \bar{C}$. [LTR, see hw]
But $\bar{C} \cap D = \emptyset$ [def. connectedness]
So $\bar{A} \cap D = \emptyset$.
Since $C \cup D \subseteq \bar{A}$, then $D \subseteq \bar{A} \to D = \emptyset$ ※ [separations nonempty] □

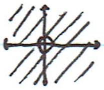
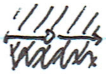
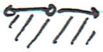
Ex: Prop: S^1 is connected [proved in one way in L11 hw]

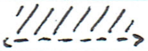
Pf: recall $S^1 \setminus \{i\} \cong \mathbb{R}$

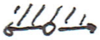
Since $\mathbb{R}$ connected, $S^1 \setminus \{i\}$ connected.

Then $S^1 \setminus \{i\} \subseteq \overline{S^1 \setminus \{i\}} = S^1$, so by prev. thm, S^1 connected.

Ex: Consider $\mathbb{R}^2 \setminus \{(x_0, y_0)\}$ (wlog, (x_0, y_0) is origin).

↳  =  ∪  is connected (p

o note:  = $\mathbb{R} \times (0, \infty)$ is connected (first thm)

↳ so  = $\mathbb{R} \times (0, \infty) \cup$ {some limit pts} is connected (prev. thm)

Thm: $\mathbb{R} \not\cong \mathbb{R}^2$

Pf: (sketch) Suppose $\mathbb{R} \cong \mathbb{R}^2$

Then $\exists$ homeomorphism $f : \mathbb{R} \hookrightarrow \mathbb{R}^2$ by $0 \mapsto f(0) = (a, b)$

So $\bar{f} : \mathbb{R} \setminus \{0\} \hookrightarrow \mathbb{R}^2 \setminus \{(a, b)\}$ by $x \mapsto f(x)$ is a homeomorphism.

But $\mathbb{R} \setminus \{0\}$ is not connected, while $\mathbb{R}^2 \setminus \{(a, b)\}$ is. $\divideontimes$

Problems for Lesson 12: Connectedness as a Topological Invariant

February 13, 2017

Problem (2) will be graded.

(1) Prove that if $A \subseteq C$ in the space X. Then $\overline{A} \subseteq \overline{C}$.

Comment: This is a step used in today's proof that if A is connected in X and $A \subseteq B \subseteq \overline{A}$, then B is also connected.

(2) Prove that S^2 is connected, where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \big| x^2 + y^2 + z^2 = 1\}$ is the unit sphere in the 3D Euclidean space.

Hint: There are more than one method. For example, you can view S^2 as the union of the closed northern hemisphere and the closed southern hemisphere, each of which is the homeomorphic image of the closed unit disk on $\mathbb{R}^2$. Alternatively, you can view S^2 as the closure of $S^2 \backslash \{(0, 0, 1)\}$.

(3) Prove **The Intermediate Value Theorem.** Let X be a connected space and $f : X \to \mathbb{R}$ be continous. If $a, b \in f(X)$ and c satisfies $a < c < b$, then there is $x \in X$ such that $f(x) = c$.

Hint: Suppose this is not the case, namely $c \notin f(X)$. Then $(-\infty, c) \cap f(X)$ and $(c, \infty) \cap f(X)$ form a separation of $f(X)$. (You do need to check this). This contradicts the fact that $f(X)$ should be connected.

(4) Prove **The One-Dimensional Brouwer Fixed Point Theorem.** Any continuous function $f : [-1, 1] \to [-1, 1]$ has a fixed point (there is $x_0 \in [-1, 1]$ such that $f(x_0) = x_0$.)

Hint: Consider the function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$, which is continuous. Apply the Intermediate Value Theorem.

(5) Why doesn't method of the last proof we did in class today work for proving that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$?

2/15 Path-Connectedness

Def: Let X be a space. A path in X is a $\overset{\text{(cont. fxn)}}{\text{map}}$ $\gamma : \underset{I}{[0,1]} \to X$

- we say $\gamma(0)$ and $\gamma(1)$ are joined by γ
- $\gamma^{-1}(t) := \gamma(1-t)$ [note: abuse of notation: $\gamma^{-1} : [0,1] \to X$ still]
  - note: γ^{-1} could also mean inverse image operation (e.g. $\gamma^{-1}A$ for $A \subseteq X$)

Def: To join $\alpha : I \to X$ and $\beta : I \to X$, define $\gamma : I \to X$ by $t \longmapsto \alpha(2t)$ $0 \le t \le \frac{1}{2}$

$t \longmapsto \beta(2t-1)$ $\frac{1}{2} \le t \le 1$

- note: I still goes from $0 \to 1$.

Lemma: (Pasting Lemma) This $\gamma : I \to X$ is continuous, i.e. still a path

Pf: LTR (homework)

Def: X is path-connected if any two points in X can be joined by a path in X.

- i.e. $\forall x, y \in X$, $\exists$ path $\gamma : I \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$
- e.g. $\mathbb{R}^2, S^1, S^2$ are path-connected

Note: path-connected $\Rightarrow$ connected always

↳ connected $\Rightarrow$ path-connected sometimes (when local path-connected)

1 Thm: If X is path-connected, then X is connected.

Pf: Let $A \subseteq X$, $A \neq \emptyset$, A both open and closed in X.

[goal: show $A = \emptyset$ or $X \Rightarrow X$ connected]

Suppose $A \neq X$, i.e. $X \setminus A \neq \emptyset$

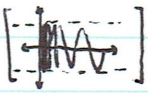
Let $x \in A$, let $y \in X \setminus A$, let $\gamma : I \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$ [X path-connected]

Since A both open and closed in X, $\gamma^{-1}A$ both open and closed in I.

Since $y \notin A$, $1 \notin \gamma^{-1}A$, so $\gamma^{-1}A \neq [0,1]$

Yet $0 \in \gamma^{-1}A$, so $\gamma^{-1}A \neq \emptyset$

But $[0,1]$ is connected ⨳ [L11, T②]



Def: The Topologist's Sine Curve is $\{0\} \times [-1,1] \cup \{(x, \sin\frac{1}{x}) \mid 0 < x < 2\pi\}$

Prop: TSC is connected, but not path-connected.

<u>Pf</u>: see textbook figure 3.4

2 Thm: A connected open subset of $\mathbb{R}^n$ is path-connected.

<u>Pf</u>: (sketch) If $A = \emptyset$, vacuously true.

Suppose $A \neq \emptyset$. Let $x \in A$.

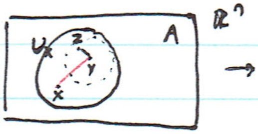
Define $U_x = \{y \in A \mid x, y$ can be joined by path in $A\}$

Then U_x is path-connected [on homework]

We show that $U_x = A$ [by showing U_x open, $A \setminus U_x$ open]

Let $y \in U_x \subseteq A$, i.e. $y \in A$. (open).

So $\exists\, B_\varepsilon(y)$ in $\mathbb{R}^n$ s.t. $y \in B_\varepsilon(y) \subseteq A$.



→ use natural path b/w $z \in B_\varepsilon(y)$ and y, pasted w/ path b/w x and y → $B_\varepsilon(y) \subseteq U_x \Rightarrow U_x$ open.

Similarly, $A \setminus U_x$ is open.

So U_x closed.

So $U_x = A$ [L11, T③]

Problems for Lesson 13: Path-Connectedness

February 15, 2017

Problem (5) will be graded.

(1) **The Pasting Lemma.** Let A and B be closed subsets of the space X and $A \cup B = X$. If $f : A \to Y$ and $g : B \to Y$ are continuous functions and they agree over $A \cap B$, namely $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is also a continuous function.

Hint: Use Problem (1) of L4, the set-theoretic fact $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, the fact that a closed set in a subspace of a space X equals the intersection of a closed set of X with the subset, and Problem (1) of L2.

Comment: The pasting lemma can be used to prove that the join of two paths $f : [0,1] \to X$, $g : [0,1] \to X$ is a path. Before you apply it, you also need to precompose the continuous functions $l : [0,1/2] \to [0,1]$ sending $t \mapsto 2t$ and $r : [1/2,1] \to [0,1]$ sending $t \mapsto 2t-1$ to f and g respectively and use the fact that the composite of continuous functions is continuous.

(2) Let X be a space and $x_0 \in X$. Show that the space

$$U_{x_0} := \{y \in X \big| x_0 \text{ and } y \text{ are joined by a path in } X\}$$

is path connected.

Hint: Use (1). Given $x, y \in U_{x_0}$, each of x and y can be joined by a path to x_0. You can combine these two path to get a path joining x and y.

(3) Let A and B be path-connected subsets of space X and $A \cap B$ is nonempty. Prove that $A \cup B$ is also path-connected.

Hint: Similar to the above.

(4) Prove that the continuous image of a path-connected space is path-connected.

(5) Prove that the product of two path-connected spaces is path-connected.

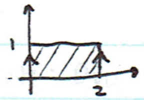
Hint: Connect two general points using an "L"-shaped path.

(6) Read the proof of (3.30) in the textbook, which we sketched in class. Where does the proof break down if we remove the condition that the connected set is **open**?

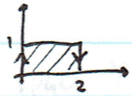
2/16 Quotient / Identification Spaces

Intuition:

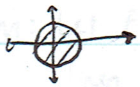
- in $\mathbb{R}^2$: we produce a cylinder by "gluing" $(0,y)$ and $(2,y)$ $\forall y \in [0,1]$



- in $\mathbb{R}^2$: we produce a Möbius strip by "gluing" $(0,y)$ and $(2,1-y)$



- in $\mathbb{R}^2$: we produce a sphere by "pinching" the boundary into 1 point.



- in $\mathbb{R}^3$: we produce a sphere by "pinching"  into 1 point

→ how to: 1) define "glue", "pinch", etc. in a mathematical way? (set)
2) define a topology on this new set? (topology)

Def: Let X be a space. $\{A_i \mid i \in I\}$ is a partition of X if:
① $A_i \neq \emptyset$ $\forall i \in I$ (nonempty)
② $A_i \cap A_j = \emptyset$ $\forall i \neq j \in I$ (pairwise disjoint)
③ $\bigcup_{i \in I} A_i = X$

- there is a natural surjection $X \to \{A_i \mid i \in I\}$ by $x \mapsto \overset{②}{\text{unique}}\ A_i\ \overset{③}{\text{containing}}\ x$

Ex: $X = \{1,2,3,4,5\} \to \{\{1,2\},\{3,4\},\{5\}\}$

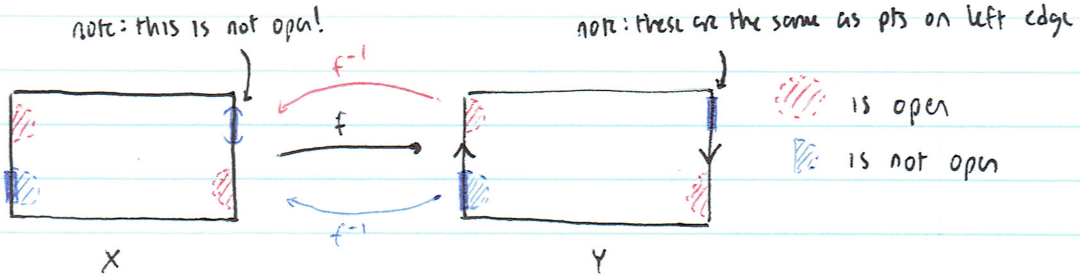
Ex: $X = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2, 0 \le y \le 1\} \to \{\{(x,y)\} \mid 0 < x < 2, 0 \le y \le 1\} \cup$
$\{\underbrace{\{(0,y),(2,1-y)\}} \mid 0 \le y \le 1\}$
↳ note: this is a single point! "gluing"

Def: Let X be a space. Let Y be a set (e.g. $Y = \{A_i \mid i \in I\}$). Let $f: X \to Y$ be a surjection. We define the quotient/identification topology on Y as follows: A set O in Y is open $\Leftrightarrow$ $f^{-1}(O)$ is open in X.

- note: by definition, $f: X \to Y$ is automatically continuous.
- note: iff $\Rightarrow$ quotient topology of Y consists of all subsets of Y whose inverse image is open in X (i.e. if $f^{-1}(O)$ is open in X, O is in top. on Y)
  ↳ so the quotient topology on Y is the largest one s.t. $f: X \to Y$ is continuous

Pf: LTR (homework)

Ex:



Note: Due to the nature of quotient space Y, mappings out of Y are best studied by studying mappings out of X.

Thm: Let $f: X \to Y$ be a quotient map. Let Z be another space. Then a function $g: Y \to Z$ is continuous <=> the composite function $g \circ f: X \to Z$ is continuous.

∘ intuition:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

(these digrams are very useful)

Pf: (=>) Suppose $g: Y \to Z$ is continuous.
Since $f: X \to Y$ is a quotient map, f is continuous.
So $g \circ f$ is continuous.

(<=) Suppose $g \circ f: X \to Z$ is continuous.
Let O be open in Z.
Then $(g \circ f)^{-1}(O)$ is open in X [$g \circ f$ continuous]
Since $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is open,
and $g^{-1}(O)$ is open in Y, [f is quotient map]
then g is continuous. □

Problems for Lesson 14: Quotient/Identification Spaces

February 16, 2017

Problem (2) will be graded.

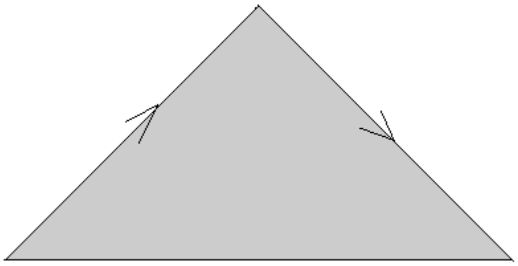
(1) Prove that the quotient topology indeed is a topology.

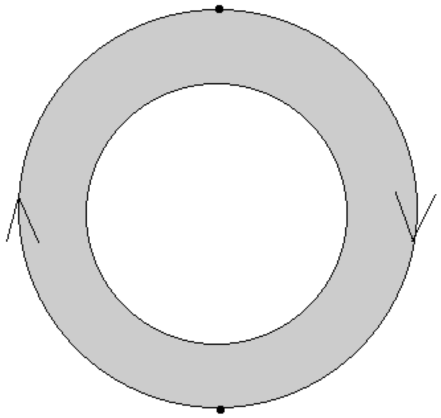
(2) Prove again the last theorem we proved in class today.

Comment: This is an important theorem. It will be used tomorrow.

(3) The following picture shows the standard way to obtain the Möbius strip via a quotient/identification process.

Below are two other ways to obtain the Möbius strip. Try to see why that's the case.





Hint: If it is too difficult to imagine these two spaces in three dimensional space, then you can try cutting each into two simpler pieces and then reassemble them in a different way. Your Beginning Topologist's Toolbox might be of some help.

2/17 Maps out of quotient spaces

Motivation: to show 'mathematical gluing' from yesterday is 'physical gluing'

∘ e.g. 

Recall L7, T3: if $g(Y) \to Z$ is a continuous bijection where Y is compact, Z is Hausdorff, then g is a homeomorphism

Recall intuition from L14:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

when f is a quotient map

∘ and last thm from L14: $g \circ f$ continuous $\Leftrightarrow$ g continuous

Fact: X compact $\Rightarrow$ Y compact

Pf: $Y = f(X)$ for compact X

Thm: (The Main Theorem)

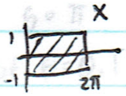
④ X is compact

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "\text{③ } g\circ f \text{ continuous}"] & \\ Y \arrow[r, "\text{① } g \text{ is well-defined}", "\text{② } g \text{ is 1:1 and onto}"'] & Z \end{tikzcd}$$

⑤ Z is Hausdorff

$\Rightarrow g: Y \to Z$ is a homeomorphism

Pf: LTR (on homework, using facts above)



Ex: Let $X = \{(\theta,t) \in \mathbb{R}^2 \mid 0 \le \theta \le 2\pi, -1 \le t \le 1\}$

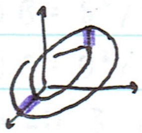
$Y = \{ \{(0,t),(2\pi,-t)\} \mid -1 \le t \le 1\} \cup \{\{(\theta,t)\} \mid 0 < \theta < 2\pi, -1 \le t \le 1\}$

Let $f: X \to Y$ by $(\theta,t) \mapsto \{(\theta,t)\}$ $0 < \theta < 2\pi$, $-1 \le t \le 1$

and $(0,t),(2\pi,-t) \mapsto \{(0,t),(2\pi,-t)\}$

Ex: (cont'd) Let $Z = \{(x,y,z) \in \mathbb{R}^3 \mid x = \underbrace{(5+t\cos\frac{\theta}{2})\cos\theta}_{a(\theta,t)}$, $y = \underbrace{(5+t\cos\frac{\theta}{2})\sin\theta}_{b(\theta,t)}$, $z = \underbrace{t\sin\frac{\theta}{2}}_{c(\theta,t)}$, $0 \le \theta \le 2\pi$, $-1 \le t \le 1$

- then this is a Möbius strip:



Define $g: \{(\theta,t)\} \mapsto (a(\theta,t), b(\theta,t), c(\theta,t))$ if $0 < \theta < 2\pi, -1 \le t \le 1$

$$\{(0,t),(2\pi,-t)\} \mapsto \left(\underset{\overset{\shortparallel}{a(2\pi,t)}}{a(0,t)},\ \underset{\overset{\shortparallel}{b(2\pi,t)}}{b(0,t)},\ \underset{\overset{\shortparallel}{c(2\pi,t)}}{c(0,t)}\right)$$

So: ① g is well-defined.

Clearly, ② g is 1:1 and onto.

And ③ $g \circ f: X \to Z$ $(\theta,t) \mapsto (a(\theta,t), b(\theta,t), c(\theta,t))$ is continuous

__Pf__ : calculus fact - composition of continuous functions is continuous

④ X is compact (closed and bounded in $\mathbb{R}^2$)

⑤ Z is Hausdorff (because Z is subspace of Hausdorff space $\mathbb{R}^2$)

$\Longrightarrow X \cong Y$

Ex: Given space X, let $A \subseteq X$. Then X/A is the set obtained by identifying all points in A into a single point, i.e. $\{\{x\} \mid x \in X \setminus A\} \cup \{A\}$

Prove that $[0,1]/\{0,1\} \cong S^1$

__Pf__ :

$$\begin{tikzcd} {[0,1]} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ {[0,1]/\{0,1\}} \arrow[r, "g"'] & S^1 \end{tikzcd}$$

$f: [0,1] \to S^1$ by $\theta \mapsto e^{i2\pi\theta}$

$g: [0,1]/\{0,1\} \to S^1$ by $\theta \mapsto 2^{i2\pi\theta}$ if $0 < \theta < 1$

$\{0,1\} \mapsto e^{i2\pi 0} = e^{i2\pi \cdot 1}$ ow.

Notice that $f =$ ~~$g \circ f$~~ $g \circ \pi$

① g is well-defined (i.e. both representations of $\theta = 0,1$ go to same point)

② g is a bijection

③ $f = g \circ \pi$ is continuous

④ $[0,1]$ is compact (closed & bounded in $\mathbb{R}$)

⑤ S^1 is Hausdorff ($\mathbb{R}^2$ is Hausdorff)

$\Longrightarrow [0,1]/\{0,1\} \cong S^1$

Ex: $\mathbb{R}P^2$ (the real projective space of dimension n) can [redacted] be defined as:

① $A = \mathbb{R}^3 \setminus \{(0,0,0)\}$ / "id each straight line thru origin into single point"

② $B = S^2 / x \sim -x$

③ $C = D^2$ / "id each pair of antipodal points on boundary (S^1)"

↳ all 3 characterizations are the same

Pf: LTR (see homework)

(sketch)

$(x,y) \mapsto (x,y,\sqrt{1-x^2-y^2})$ $\quad$ $(x,y,z) \mapsto (x,y,z)$

$$\begin{tikzcd}
D^2 \arrow[r, hook, "f_2"] \arrow[d, "\pi_C"'] \arrow[dr, "\pi_B \circ f_2"'] & S^2 \arrow[r, hook, "f_1"] \arrow[d, "\pi_B"'] \arrow[dr, "\pi_A \circ f_1"'] & \mathbb{R}^3 \setminus \{(0,0,0)\} \arrow[d, "\pi_A"'] \\
C \arrow[r, "\text{define}"'] & B \arrow[r, "\text{define}"'] & A
\end{tikzcd}$$

(Hausdorff)

↳ "pf":

Problems for Lesson 15: Maps out of quotient spaces

February 17, 2017

Problem (3) will be graded.

1. Prove again **The Main Theorem** we stated and proved in class today.

2. Let D^n be the unit closed ball in $\mathbb{R}^n$ and S^{n-1} the boundary sphere of D^n. Prove that D^n/S^{n-1} is homeomorphic to S^{n+1}.

   Hint: You can read the top half on Page 69 to get some idea. But keep in mind that the method used in the book is slightly different from ours, though they are fundamentally the same.

3. Prove that the three ways of defining the real projective n space $\mathbb{R}P^n$ yield homeomorphic spaces.

   Hint: Follow the diagram outlined in class. Prove from the right to the left. (When checking continuity for a map in The Main Theorem, just say it's continuous.)

MATH 455 Quiz #4

Name:____________________

1. (2 points) Let X be a space, Y a set and $f : X \to Y$ a function from the set X **onto** the set Y. Define the quotient topology on Y (induced from this f).

For the next four problems, just write T or F. You don't have to explain.

2. (2 points) True or False? The space obtained from the closed unit disk in $\mathbb{R}^2$ by collapsing the boundary circle into a single point is homeomorphic to S^2.

3. (2 points) True or False? If a space is connected, then it is also path-connected.

4. (2 points) True or False? If X and Y are connected, then so is $X \times Y$.

5. (2 points) True or False? If X and Y are path-connected, then so is $X \times Y$.

2/20 More about the real projective plane ($\mathbb{RP}^2$)

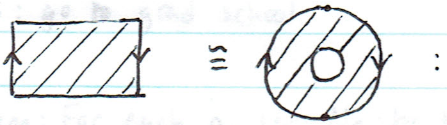
Recall: 3 ways to define $\mathbb{RP}^2$ in L15

- 4th way: glue Möbius strip to D^2 along boundary circles
  - note: Möbius strip has a single boundary circle!

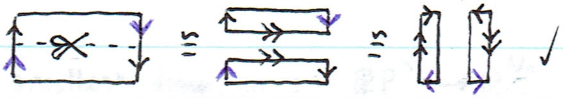
Thm: Let X, Y be disjoint spaces. Then we can define a topology on $X+Y$ ("disjoint union") as follows: O is open in $X+Y$ if $O = O_1 \cup O_2$ where O_1 open in X, O_2 open in Y.

Pf: LTR.

Recall: L14, P3:

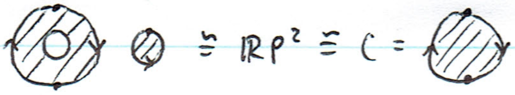


why? consider redrawing the LHS picture:

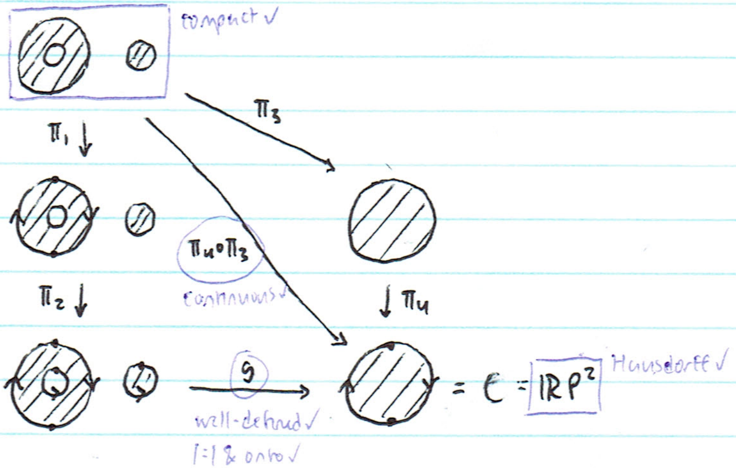


- note: this could be proved rigorously (via Main Theorem)

⟶ so we can reformulate 4th def. of $\mathbb{RP}^2$ as:



Pf:



~~[illegible]~~

Def: $f : X \to Y$ is an embedding if f is a homeomorphism from X onto its image $f(X)$

↖ w/ subspace top. in Y

Thm: $\mathbb{RP}^2$ embeds in $\mathbb{R}^4$

Pf: (outline)

↙ by $(x,y,z) \mapsto (x^2-y^2, xy, yz, zx)$

$$\begin{tikzcd} S^2 \arrow[d, "\pi_B"'] \arrow[dr] & \\ \mathbb{RP}^2 \arrow[r, "g"] & \mathbb{R}^4 \end{tikzcd}$$

↑

by $\{(x,y,z),(-x,-y,-z)\} \mapsto (x^2-y^2, xy, yz, zx)$

Thm: $\mathbb{RP}^2$ does not embed in $\mathbb{R}^3$

Pf: go to grad school.

Thm: For each n, let N_n be the smallest dimension s.t. $\mathbb{RP}^n \hookrightarrow \mathbb{R}^{N_n}$.
What are all these N_n?

Pf: unknown - go get some Fields medals

Problems for Lesson 16: More about the Real Projective Spaces

February 20, 2017

Problem (3) will be graded.

(1) Let $X + Y$ be the disjoint union of two given disjoint spaces X and Y. Recall that a subset O is open in $X + Y$ if and only if $O = O_1 \cup O_2$ where O_1 is open in X and O_2 is open in Y. Prove that this indeed gives a topology on $X + Y$.

(2) Prove that if X and Y are disjoint compact spaces, then $X + Y$ is also compact.

(3) Prove that $S^2 \longrightarrow \mathbb{R}^4$ defined by $(x, y, z) \mapsto (x^2 - y^2, xy, yz, zx)$ induces an embedding from $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Comment: You can directly cite the fact that the induced map on the bottom of the triangular diagram is injective, though it is much more fun checking this fact on your own, partially because you would discover that $(x, y, z) \mapsto (xy, yz, zx)$ alone doesn't induce an injective map from $\mathbb{R}P^2$ into $\mathbb{R}^3$. It's a fact that $\mathbb{R}P^2$ does not embed into $\mathbb{R}^3$.

2/20 Topological Groups

Def: A group is a set G with a $\overset{(\text{"binary operation"})}{\text{function}}$ $\underset{(g_1,g_2)\mapsto g_1g_2}{m: G\times G \to G}$ satisfying:

① Associativity: $g_1(g_2g_3) = (g_1g_2)g_3$

② Identity: $\exists e \in G$ s.t. $eg = g = ge \quad \forall g \in G$.

③ Inverse: $\forall g \in G$, $\exists g^{-1} \in G$ s.t. $gg^{-1} = e = g^{-1}g$.

∘ note: this means there is a function $i: G \to G$ by $g \mapsto g^{-1}$

Note: identity, inverses are unique

Ex: $(\underset{\overset{\nearrow}{\text{set}}}{\mathbb{R}}, \underset{\overset{\uparrow}{m}}{+}, \underset{\overset{\uparrow}{e}}{0})$, $(\mathbb{R}_{>0}, \cdot, 1)$, $(\underset{\overset{\nearrow}{\text{symmetric gp.}}}{S_n}, \underset{\overset{\nwarrow}{\text{fxn composition}}}{\circ}, id_n)$

Def: A topological group G is:

① A topological space

② A group

③ $m: G\times G \to G$ by $(g_1, g_2) \mapsto g_1g_2$, $i: G \to G$ by $g \mapsto g^{-1}$ are continuous.

Note: in textbook, G is also required to be Hausdorff

Ex: $(\mathbb{R}, +, 0)$ is a topological gp. (w/ standard top.)

Pf: clearly, ① and ② hold.

$m: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by $(x,y) \mapsto x+y$ is continuous [calculus fact]

$i: \mathbb{R} \to \mathbb{R}$ by $x \mapsto -x$ is continuous

Ex: similarly, $(\mathbb{R}_{>0}, \cdot, 1)$ is a topological gp. (w/ subspace top.)

Ex: $(S_n, \circ, id_n)$ w/ discrete topology is a topological gp. (trivially)

Def: An isomorphism between topological groups G and H is a homeomorphism $f: G \to H$ which is also a group homomorphism

- i.e. f is a bijection, f is a homomorphism $\implies$ f is an isomorphism.
- recall: homomorphisms preserve gp structure: $f(g_1 g_2) = f(g_1) f(g_2)$

Ex: $f: \mathbb{R} \to \mathbb{R}_{>0}$ by $x \mapsto e^x$, $\ln y \leftarrow\!\shortmid\, y$ is an isomorphism

Pf: f is homeomorphism [e^x, $\ln y$ cont.]

$$f(x+y) = e^{x+y} = e^x e^y = f(x) f(y) \quad \square$$

Def: A topological subgroup H of G is a subgroup of G and a subspace of G.

Ex: $(\mathbb{Z}, +)$ is top. subgp. of $(\mathbb{R}, +)$

Def: H is a topological normal subgroup of G if:

① H is a topological group

② H is normal

- recall: $gHg^{-1} \subseteq H \;\; \forall g \in G$.

Def: A connected component of space X is a connected subset C of X s.t. if C' is connected in X s.t. $C \subseteq C'$, then $C = C'$.

- i.e. C is a maximal connected subspace of X.

Thm: Let G be a topological group. Let K be a connected component of G s.t. $e \in K$. Then K is a closed normal (topological) subgroup.

Pf: (see L18)

- Def: (left/right translations) Let $g \in G$. Then $L_g: G \to G$ by $x \mapsto gx$ is a homeomorphism.
  - note: $L_g^{-1} = L_{g^{-1}}$
  - L_g is a homeomorphism

  Pf: $L_x: G \to G \times G \to G$ by $y \xmapsto{(\text{constant } x,\ id)} (x, y) \xmapsto{m} xy$ is continuous [comp. of cont.]

  (we can let $x = g$, $x = g^{-1}$)

Problems for Lesson 17: Topological Groups

February 22, 2017

Problem (1) will be graded.

(1) Let $\mathbb{R}^2$ be equipped with the standard topology. Define the first binary operation as $(x, y) \oplus (x', y') = (x + x', y + y')$. Define the second binary operation as $(x, y) \odot (x', y') = (x + x'e^{-y}, y + y')$.
  - Show that $(\mathbb{R}^2, \oplus)$ is a topological group.
  - Show that $(\mathbb{R}^2, \odot)$ is also a topological group.
  - Show that these two topological groups are not isomorphic. *Hint:* You can directly use the fact that for two isomorphic groups, if one is abelian, then so is the other.

(2) Given a space X and a connected subset C, recall that C is called a **connected component** if for any connected subset C' of X with the property that $C \subseteq C'$, then $C = C'$. This means C is actually a maximal connected subset of X (because if there is a potentially bigger connected subset C', then C' has to be C).

  - Prove that C is closed.
    Hint: This is the first half of Theorem 3.27 in the textbook. Or just use the fact we proved in class that if A is connected and $A \subseteq B \subseteq \overline{A}$, then B is also connected. (Let $B = \overline{A}$.)
  - Prove that every connected subset of space is contained in a connected component.
    Hint: The proof is the paragraph after Theorem 3.27 in the textbook. Or you can prove it on your own using Problem (3) of L11.

(3) Prove that if H is a topological subgroup of G, then $\overline{H}$ is also a topological subgroup of G. Furthermore, if H is normal in G, then $\overline{H}$ is also normal in G.

Hint: A tedious problem, but everything follows from definition.

2/23 Thm: Let G be a topological group. Let K be a connected ~~subset~~ component of G, $e \in K$. Then K is a: ① closed, ② normal ③ subgroup.

Pf: (①) Since K is a connected component, K is closed [L17, P2]

(③) $\forall x \in K$, recall $R_{x^{-1}} : G \to G$ by $g \mapsto gx^{-1}$ is a homeomorphism.

So $Kx^{-1} = R_{x^{-1}}(K)$ is connected [cont. map]

Since $x \in K$, $xx^{-1} \in R_{x^{-1}}(K)$, i.e. $e \in R_{x^{-1}}(K)$

So $R_{x^{-1}}(K) \subseteq K$ [L17, P2]

But $K_{x^{-1}} = R_{x^{-1}}(K) \subseteq K$ (so $\forall x, y \in K$, $yx^{-1} \in K$)

(②) $\forall g \in G$, recall $L_g : G \to G$ by $x \mapsto gx$, $R_{g^{-1}} : G \to G$ by $x \mapsto xg^{-1}$ are homeomorphisms

So $gKg^{-1} = R_{g^{-1}}(L_g(K))$ is connected [$R_{g^{-1}}(L_g)$ cont.] [K conn.]

Notice that $e = geg^{-1} \in gKg^{-1}$

By reasoning similar to above, $gKg^{-1} \subseteq K$ [K conn. comp. $\ni e$]

L18

2/23 Matrix Groups

Def: $M(n)$ = the set of all $n \times n$ matrices with real entries

- note: $M(n)$ is the same as $\mathbb{R}^{n^2}$

$$\underset{(n\times n)}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}} \longmapsto \underset{(n^2\times 1)}{\begin{bmatrix} a_{11} \\ \vdots \\ a_{nn} \end{bmatrix}}$$

- note: we could use matrix addition to form group $\left(e = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}\right)$
  - ↳ but this is not interesting.
- note: using matrix multiplication is more interesting
  - ↳ but not every $n \times n$ matrix is invertible...

Def: The general linear group $GL(n)$ is the subspace of $M(n)$ consisting of all invertible $n \times n$ matrices

- i.e. $GL(n) = \det^{-1}(\mathbb{R} \setminus \{0\})$
  - note: $\det : M(n) \to \mathbb{R}$ by $A \mapsto \det A$ is continuous [it's a polynomial]
  - note: since $\mathbb{R} \setminus \{0\}$ is open, det cont, $GL(n)$ is open in $\mathbb{R}^{n^2}$
    - ↳ so $GL(n)$ is not compact.

Prop: $GL(n)$ is not connected

Pf: $\det^{-1}(-\infty, 0)$ and $\det^{-1}(0, \infty)$ form a separation of $GL(n)$

- note: $\det^{-1}(-\infty, 0)$, $\det^{-1}(0, \infty)$ are both connected components.

Thm: $GL(n)$ is a topological group

Pf: $GL(n)$ is a topological space [subspace of $M(n)$ "=" $\mathbb{R}^{n^2}$]

$GL(n)$ is a group [identity $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$]

$m : GL(n) \times GL(n) \to GL(n)$ by $([a_{ij}], [b_{ij}]) \mapsto [c_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$ is a polynomial, so it's continuous.

$i : GL(n) \to GL(n)$ by $[a_{ij}] \mapsto [b_{ij}] = [\frac{1}{\det[a_{ij}]} \cdot \text{cofactor of } a_{ij}]$ is a rational function (i.e. quotient of polynomials), so it's continuous where it's defined, i.e. everywhere □

Def: The <u>orthogonal group</u> $O(n) = \{ A \in M(n) \mid AA^T = I_n \}$

- e.g.:

$$\underset{A}{\begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{bmatrix}} \underset{A^T}{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ so $O(n)$ contains $n \times n$ matrices whose rows are mutually orthogonal vectors of unit length.

- note: $O(n)$ is a subgroup of $GL(n)$ [i.e. $O(n) < GL(n)$]

  PF: note that $(\det A)^2 = \det A \det A^T = \det AA^T = \det I_n = 1$

  so $\det A = \pm 1$, i.e. $\det A \neq 0$.

Def: The <u>special orthogonal group</u> $SO(n) = \{ A \in O(n) \mid \det A = 1 \}$

- note: this models rotations
  - if $\det = -1$, we get mirrored reflections

Problems for Lesson 18: Matrix Groups

February 23, 2017

Problem (1) will be graded.

(1) Recall that the orthogonal matrix group $O(n)$ and special orthogonal matrix group $SO(n)$ are defined as follows:

$$O(n) = \{A \in M(n) \big| AA^T = I_n\}, SO(n) = \{A \in O(n) \big| \det(A) = 1\}.$$

  (a) Prove that $O(n)$ and $SO(n)$ are topological subgroups of the general linear group $GL(n)$.

  (b) Prove that $O(n)$ and $SO(n)$ are compact. *Hint:* This is Theorem 4.13 in the textbook.

(2) The special linear group $SL(n)$ is defined as $SL(n) = \{A \in M(n) \big| \det(A) = 1\}$. Prove that $SL(n)$ is a topological subgroup of $GL(n)$.

Exam 1 Study Guide

Exam 1 will take place on **Friday, March 3th**, in our regular classroom **Seeley Mudd 207** during our regular class time from **11:00 A.M. to 11:50 A.M.** It covers the material From Lesson 1 to Lesson L18. You will not be allowed to use notes, books, calculators, etc. All you need are pencils (pens) and erasers.

The exam will have five problems. Each problem is worth 10 points. Each problem may have several parts. You may be asked to state a definition, state a theorem, judge whether a statement is true or false, or prove a statement. If you are asked for a proof, you have to give a logically correct proof written in English sentences. Scratch work is not considered a proof.

Below is a list of topics from L1 to L18 which you must know for this exam. Exam problems will be similar to quiz problems, homework problems and anything we did in class. Carefully go through your notes and homework.

A practice exam will be posted in Moodle. Treat that as a real exam. Find a nice and quiet place and then try it within the 50-minute time constraint. The **solution** will also be posted in Moodle so that you know what I expect from you.

On the day before the exam (Thursday, March 2nd), I will answer your questions in an evening review session. **SMUD 206** has been reserved from **6:30 to 8:00 P.M.** for it.

- L1 Introduction
  - lots of examples of spaces, continuous maps and homotopies

- L2 Topological Spaces
  - the axioms of a topology
  - the equivalent way of defining topology using closed sets
  - lots of examples

- L3 Bases for Topological Spaces
  - basis
  - two equivalent ways of generating a topology from a basis
  - the proof that a collection of open sets is a basis generating a given topology

- L4 Continuous Functions
  - continuous function
  - the closed set characterization of continuous function
  - characterization of continuous function using basic open sets
  - subspace topology
  - equivalence between the $\epsilon - \delta$ definition of continuity and the open set definition of continuity for $f : \mathbb{R} \to \mathbb{R}$

- L5 Homeomorphism
  - basis for subspace topology
  - homeomorphism
  - homeomorphic spaces
  - Hausdorff spaces
  - the proof that Hausdorffness is preserved by homeomorphism
  - the proof that $\mathbb{R}_{fc}$ and $\mathbb{R}$ are not homeomorphic

- L6 Introduction to Compactness
  - the analogy between compactness and finiteness
  - open cover

  - finite subcover
  - compactness
  - closed set characterization of compactness
  - the continuous image of a compact space is compact
  - locally compact
  - metric space

- L7 Hausdorff Space, Compact Set, Closed Set and Homeomorphism
  - proof that a compact subset of a Hausdorff space is closed
  - proof that a closed subset of X in a compact subspace D of X is compact
  - a continuous injection whose domain is compact and whose codomain is Hausdorff is a homeomorphism onto its image
  - the union of two compact subsets is compact
  - the intersection of an arbitrary collection of compact subsets of a Hausdorff space is compact

- L8 Compact Sets in Metric Spaces
  - $[a, b]$ is closed in $\mathbb{R}$
  - proof that a compact subset in a metric space is closed and bounded
  - proof that A is compact in $\mathbb{R}$ if and only if A is closed and bounded in $\mathbb{R}$
  - If A is compact in metric space X, then any sequence in A has a subsequence converging to a point in A.
  - real-valued continuous functions defined on compact domains attains max/min
  - be aware of the Lebesgue's Lemma

- L9 One-Point Compactification
  - one-point compactification of a Hausdorff space
  - proof that the compactified space indeed is a compact topological space
  - proof that the space before compactification is indeed a subspace of the compactified space
  - the one-point compactification of a locally compact Hausdorff space is Hausdorff
  - examples of one-point compactification

- L10 Product Topological Spaces
  - In $X \times Y$, why don't we just define an open set as the product of an open set in X and an open set in Y?
  - basis for product topology
  - proof of the tube lemma
  - the product of two compact spaces is compact
  - application: a subset of $\mathbb{R}^n$ is compact if and only if it's closed and bounded

- L11 Connected Topological Spaces
  - limit point
  - closure
  - a set being closed is equivalent to the set being equal to its closure
  - connected
  - equivalent definitions
  - separation
  - the continuous image of a connected set is connected
  - if some connected sets have nonempty intersection, then their union is also connected
  - examples

- L12 Connectedness as a Topological Invariant
  - the proof that the product of two connected spaces is connected
  - so all Euclidean spaces are connected
  - If A is a connected subspace of X and $A \subseteq B \subseteq \overline{A}$, then B is also connected

  - lots of examples
  - the proof that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic
  - the intermediate value theorem
  - the one-dimensional Brouwer fixed-point theorem

- L13 Path-Connectedness
  - path
  - the pasting lemma
  - the join of two paths
  - path-connected
  - proof that if a space is path-connected, then it's connected
  - but the converse is incorrect (the topologist's sine curve is a counterexample)
  - proof that a connected open subset of a Euclidean space is path-connected
  - the union of two intersecting path-connected spaces is path-connected
  - the continuous image of a path-connected space is path-connected
  - the product of two path-connected spaces is path-connected

- L14 Quotient/Identification Spaces
  - the definition of quotient topology on Y from a surjective map $f : X \to Y$ where X is a space and Y is a set
  - the continuity of the above map after the quotient topology on Y is defined
  - the convenient way of proving a map from a quotient space is continuous

- L15 Maps out of Quotient Spaces
  - the Main Theorem and its proof
  - the proof that the Möbius strip defined by gluing two opposite edges of a rectangle is homeomorphic to the usual picture of a Möbius strip in $\mathbb{R}^3$
  - the proof that a closed interval with its two end points identified is homeomorphic to a cirlce
  - the three ways of defining $\mathbb{R}P^n$
  - the proof that they are homeomorphic

- L16 More about the Real Projective Spaces
  - Möbius strip can also be obtained by gluing each pair of antipodal points on one boundary circle of an annulus
  - proof that the real projective plane can also be obtained by gluing a closed disc and a Möbius strip along their boundary circles
  - embedding
  - proof that $\mathbb{R}P^2$ embeds in $\mathbb{R}^4$
  - be aware that $\mathbb{R}P^2$ does not embed in $\mathbb{R}^3$

- L17 Topological Groups
  - topological group
  - examples
  - isomorphism between topological group
  - topological subgroup
  - left and right translation as homeomorphisms
  - connected component
  - connected component is closed
  - if a connected set intersects with a connected component, then it is contained in that connected component

- L18 Matrix Groups
  - the connected component of a topological group containing the identity is a closed normal subgroup

Math 455 Topology, Spring 2017
Practice Exam 1
March 3

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) If $f : X \to Y$ is continuous and X is connected, then Y is also connected.

   (b) (2 points) $\mathbb{R}^3 \backslash \{(0,0,0)\}$ is both connected and path-connected.

   (c) (2 points) The one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.

   (d) (2 points) If $f : X \to Y$ is continuous and X is Hausdorff, then Y is also Hausdorff.

   (e) (2 points) If C_1 and C_2 are compact subsets of X, then $C_1 \cup C_2$ is also compact.

2. (10 points)

   (a) (5 points) Use definition to prove that the subspace $\{0\} \cup \{1/n \big| n = 1, 2, 3, \cdots\}$ of $\mathbb{R}$ is compact.

   (b) (5 points) Prove that the one-point compactification Y of a Hausdorff space X indeed is compact.

3. (10 points) Prove that if X is path-connected, then it is connected.

4. (10 points) Prove that $\mathbb{R}P^1$, defined as the quotient space obtained from S^1 by identifying each pair of antipodal points, is homeomorphic to S^1. State precisely the theorem(s) you use.

5. (10 points)

   (a) (5 points) Prove that under matrix multiplication, $GL(2)$, the space of all 2 by 2 invertible real matrices, is a topological group.

   (b) (5 points) Let x be an element in the topological group G. Prove that $f : G \to G$ defined by $f(g) = xgx^{-1}$ is an isomorphism between topological groups.

Math 455 Topology, Spring 2017
Practice Exam 1
March 3

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) If $f : X \to Y$ is continuous and X is connected, then Y is also connected.

   F

   (b) (2 points) $\mathbb{R}^3 \backslash \{(0,0,0)\}$ is both connected and path-connected.

   T

   (c) (2 points) The one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.

   T

   (d) (2 points) If $f : X \to Y$ is continuous and X is Hausdorff, then Y is also Hausdorff.

   F

   (e) (2 points) If C_1 and C_2 are compact subsets of X, then $C_1 \cup C_2$ is also compact.

   T

2. (10 points)

(a) (5 points) Use definition to prove that the subspace $\underbrace{\{0\} \cup \{1/n | n = 1, 2, 3,}_{A} \cdots\}$ of $\mathbb{R}$ is compact.

Proof: Let $\{O_i | i \in I\}$ be an open cover of A by open sets in $\mathbb{R}$.

Since $0 \in A$, there is O_i such that $0 \in O_i$. But O_i is open in $\mathbb{R}$, so there is $\varepsilon > 0$ such that $0 \in (-\varepsilon, \varepsilon) \subseteq O_i$.

Let $N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$. So $N > \frac{1}{\varepsilon}$. If $n \geq N$, then $0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ and thus $\frac{1}{n} \in (-\varepsilon, \varepsilon) \subseteq O_i$.

For each $\frac{1}{j}$ ~~For each~~ of $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{N-1}$, since it's in A, there is O_{i_j} in the cover such that $\frac{1}{j} \in O_{i_j}$.

Therefore, $\{O_i, O_{i_1}, O_{i_2}, \cdots, O_{i_{N-1}}\}$ is a finite subcover of $\{O_i | i \in I\}$ □

(b) (5 points) Prove that the one-point compactification Y of a Hausdorff space X indeed is compact.

Proof: We write $Y = X \cup \{\infty\}$.

Let $\{O_i | i \in I\}$ be an open cover of Y. Since $\infty \in Y$, there is O_i such that $\infty \in O_i$. So there is a compact subset C of X such that $O_i = Y \setminus C$.

Notice that since $\{O_i | i \in I\}$ covers Y and $C \subseteq Y$, $\{O_i | i \in I\}$ also covers C. But C is compact. So there are finitely many elements $O_{i_1}, \cdots, O_{i_N}$ from $\{O_i | i \in I\}$ which cover C.

Since $O_i = Y \setminus C$, we then know that $\{O_{i_1}, \cdots, O_{i_N}, O_i\}$ is a finite subcover for the space Y of $\{O_i | i \in I\}$. □

3. (10 points) Prove that if X is path-connected, then it is connected.

Proof: Suppose A, B form a separation of X.
This means ① $A, B \neq \phi$, ② $A \cap B = \phi$, ③ $A \cup B = X$ and ④ A, B are both open in X.

Since $A, B \neq \phi$, let $x \in A$, $y \in B$. Because X is path-connected, there is a path $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Let $C = \gamma^{-1}A$ and $D = \gamma^{-1}B$.

Then ① $0 \in C$, $1 \in D$. So $C, D \neq \phi$.

② Since $A \cap B = \phi$, $C \cap D = \gamma^{-1}A \cap \gamma^{-1}B$
$= \gamma^{-1}(A \cap B) = \phi$.

③ $C \cup D = \gamma^{-1}A \cup \gamma^{-1}B = \gamma^{-1}(A \cup B) = \gamma^{-1}X$
$= [0,1]$

④ Since γ is continuous and A, B are open
$C = \gamma^{-1}A$ and $D = \gamma^{-1}B$ are open in $[0,1]$.

Thus, C, D form a separation of $[0,1]$, contradicting to the fact that $[0,1]$ is connecte

Thus, there is no separation of X, showing X is connected. $\square$

4. (10 points) Prove that $\mathbb{R}P^1$, defined as the quotient space obtained from S^1 by identifying each pair of antipodal points, is homeomorphic to S^1. State precisely the theorem(s) you use.

Proof: We first cite the Main Theorem:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

(-1) $\pi: X \longrightarrow Y$ is a quotient map.
⓪ $f = g \circ \pi$
① g is well-defined
② g is 1-1, onto
③ f is continuous
④ X is compact
⑤ Z is Hausdorff.

Then Y is homeomorphic to Z.

For us, $\mathbb{R}P^1 := S^1 / \underbrace{x \sim -x}_{\text{antipodal points are identified.}}$

Define the maps as follows

$$\begin{tikzcd} S^1 \arrow[d, "\pi"'] & e^{i\theta} \arrow[d, mapsto] \arrow[dr, mapsto, "f"] & \\ & \{e^{i\theta}, -e^{i\theta}\} \arrow[r, mapsto] & e^{i2\theta} \\ \mathbb{R}P^1 \arrow[rr, "g"] & & S^1 \end{tikzcd}$$

$(e^{i\theta})^2 = e^{i2\theta}$

(-1) Obviously, π is a quotient map
and ⓪ $f = g \circ \pi$

① g is well-defined

② Since for any $e^{i\theta} \in S^1$, $g(\{e^{i\frac{\theta}{2}}, -e^{i\frac{\theta}{2}}\}) = e^{i}$

g is onto.

If $g(\{e^{i\theta_1}, -e^{i\theta_1}\}) = g(\{e^{i\theta_2}, -e^{i\theta_2}\}$

then $e^{i2\theta_1} = e^{i2\theta_2}$ So $\theta_2 = \theta_1 + (2k+1)$

Thus, $e^{i\theta_2} = \pm e^{i\theta_1}$ $\boxed{g \text{ is } 1\text{-}1.}$

So $\{e^{i\theta_1}, -e^{i\theta_1}\} = \{e^{i\theta_2}, -e^{i\theta_2}\}$

③ $f: S^1 \longrightarrow S^1$ is continuous
$z \longmapsto z^2$

④ S^1 being ~~compact~~ closed and bounded in $\mathbb{R}^2$ is compact

⑤ S^1 being subspace of Hausdorff $\mathbb{R}^2$ is Hausdorff.

Therefore, $\mathbb{R}P^1 \simeq S^1$. $\square$

5. (10 points)

(a) (5 points) Prove that under matrix multiplication, $GL(2)$, the space of all 2 by 2 invertible real matrices, is a topological group.

Proof: $GL(2) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2) \mid \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0 \}$.

We first show it's a group.

⓪ Since $\det(AB) = \det(A)\det(B)$, if $A, B \in GL(2)$, then $AB \in GL(2)$.

① matrix multiplication is associative

② $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity

③ Inverse exists.

The topology on $GL(2)$ is the subspace topology induced from $M(2)$, which is homeomorphic to $\mathbb{R}^4$.

$$m : GL(2) \times G(2) \to GL(2)$$
$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \mapsto \left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \right)$$
is continuous

polynomi are continu

$$i : GL(2) \longrightarrow GL(2)$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

rational functions an continuous.

Therefore, $GL(2)$ is a topological group.

(b) (5 points) Let x be an element in the topological group G. Prove that $f : G \to G$ defined by $f(g) = xgx^{-1}$ is an isomorphism between topological groups.

Proof: First of all, notice that $h : G \longrightarrow G$, $g \longmapsto x^{-1}gx$ is the inverse of f. So f is a bijection.

$$\begin{tikzcd} f : G \arrow[r, "L_x"] & G \arrow[r, "R_{x^{-1}}"] & G \\ g \arrow[r, mapsto] & xg \arrow[r, mapsto] & xgx^{-1} \end{tikzcd}$$

So $f = \underbrace{\underbrace{R_{x^{-1}}}_{\text{cont.}} \circ \underbrace{L_x}_{\text{cont.}}}_{\text{cont.}}$

Similarly, $h = R_x \circ L_{x^{-1}}$ is also continuous.

Thus, f is a homeomorphism.

Lastly, $f(g_1 g_2) = x g_1 g_2 x^{-1} = x g_1 x^{-1} x g_2 x^{-1}$
$= f(g_1) f(g_2)$.

So f is a group homomorphism.

Therefore, f is a ~~top~~. group isomorphism. ☐

2/24 Homotopy: motivation.

Recall: "Level 1" = spaces, "Level 2" = continuous functions between spaces
↳ "Level 3" = homotopies

Motivation: we will see the fundamental group next Monday (L20)

- idea: turn "$\{[0,1] \to X\}$" into a group.
  - we fix $x_0 \in X$ and consider paths that start & end at x_0:
    $\gamma : [0,1] \to X$ s.t. $\gamma(0) = \gamma(1) = x_0$
  - we define a binary operation: given $\alpha : I \to X$, $\beta : I \to X$ $(\alpha(0) = \alpha(1) = \beta(0) = \beta(1))$
    ↳ $\alpha \cdot \beta(s) = \{\alpha(2s)$ for $0 \le s \le \frac{1}{2}$, $\beta(2s-1)$ for $\frac{1}{2} \le s \le 1\}$
    - note: this is continuous [pasting lemma]

Def: $\underline{\Omega_{x_0} X}$:= set of all loops based at x_0 in X.

- note: $(\Omega_{x_0} X, \cdot)$ is not a gp.

Pf: Consider associativity:

$$(\alpha\cdot\beta)\cdot\gamma(s) = \begin{cases} \alpha\cdot\beta(2s) & 0 \le s \le \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} \alpha(4s) & 0 \le s \le \frac{1}{4} \\ \beta(4s-1) & \frac{1}{4} \le s \le \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

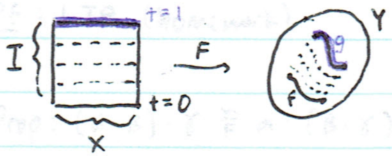
$$\alpha\cdot(\beta\cdot\gamma)(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta\cdot\gamma(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(4s-2) & \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(4s-3) & \frac{3}{4} \le s \le 1 \end{cases}$$

So even though $(\alpha\cdot\beta)\cdot\gamma$ and $\alpha\cdot(\beta\cdot\gamma)$ have the same images, they are different maps.

Def: Let $f, g : X \to Y$ be $\overset{\text{(cont. fcns)}}{\text{maps}}$. Let $I = [0,1]$. Then f is homotopic to g if $\exists$ a map $F : X \times I \to Y$ by $(x,t) \mapsto F(x,t)$ s.t. $\forall x \in X$, $F(x,0) = f(x)$, $F(x,1) = g(x)$.

- and F is a homotopy from f to g.
- note: we write $f \underset{F}{\simeq} g$

Ex: when $X = I$



Note: F interpolates between f and g.

Note: F gives us an interval many maps

- 1st is f, last is g: have maps sitting "in between" as well

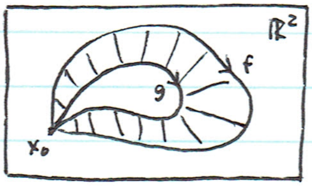
Note: we can think of F as some action that "deforms" f to g.

- i.e. "1-second movies"

Def: Let $f,g: X \to Y$ be maps. Let $A \subseteq X$. Then f is homotopic to g, relative to A, if $\exists$ a map $F: X \times I \to Y$ s.t. $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(a,t) = f(a) = g(a)$ $\forall a \in A, t \in I$.

- i.e. A is fixed.
- note: we write $f \underset{F}{\simeq} g$ rel A
- note: previous def is special case of this def ($A = \emptyset$)

Ex: Let $A = \{0,1\}$, $X = [0,I]$, $Y = \mathbb{R}^2$, $x_0 \in \mathbb{R}^2$. (note: $\forall a \in A$, $f(a) = g(a)$)



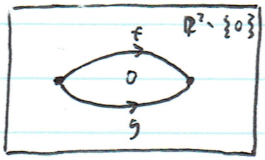
: Consider $F: X \times I \to Y$ by $(s,t) \mapsto (1-t)f(s) + tg(s)$

- note: for $s = 0$ or 1, $F = 1$.

This is called the straight-line homotopy.

Note: it's not true that any $f,g: X \to Y$ are homotopic rel A.

Ex:



: Can't deform f to g no matter what

↳ write $f \neq g$ rel $\{0,1\}$

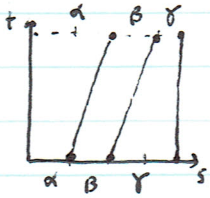
Thm: ($\simeq$ rel A) is an equivalence relation

<u>Pf</u>: LTR (homework)

Prop: $(\alpha \cdot \beta) \cdot \gamma \underset{F}{\simeq} \alpha \cdot (\beta \cdot \gamma)$ rel $\{0,1\} \subseteq I$.

<u>Pf</u>: $F : I \times I \to X$ by:

$$F(s,t) = \begin{cases} \alpha\left(\frac{4}{1+t} s\right) & 0 \leq s \leq \frac{1+t}{4} \\ \beta\left(4\left(s - \frac{1+t}{4}\right)\right) & \frac{1+t}{4} \leq s \leq \frac{2+t}{4} \\ \gamma\left(\frac{s-(2+t)/4}{1-(2+t)/4}\right) & \frac{2+t}{4} \leq s \leq 1 \end{cases} \quad \longrightarrow$$



Problems for Lesson 19: Homotopy: Motivation

February 24, 2017

Problem (2) will be graded.

(1) Prove that *being homotopic relative to a subset* $A \subseteq X$ is an equivalence relation on the set of all maps $f : X \to Y$ agreeing on A.

(2) Suppose the map $f : S^1 \to S^1$ is NOT homotopic to the identify map $id : S^1 \to S^1$. Show that $f(x) = -x$ for some $x \in S^1$.

(3) Prove that the map $f : S^1 \to S^1$ sending $x \mapsto -x$ is homotopic to the identity map $id : S^1 \to S^1$.

MATH 455 Quiz #5 Name:____________________

1. (2 points) Complete the definition: A topological group G is a space with a group structure on it such that

2. (2 points) Let $f, g : X \to Y$ be continuous functions. f is homotopic to g if

For the next three problems, just write T or F. You don't have to explain.

2. (2 points) True or False? $\mathbb{R}P^2$ can also be obtained by gluing a Möbius trip and the closed unit disk in $\mathbb{R}^2$ along their boundary circles.

3. (2 points) True or False? A connected component of a topological group G is a closed normal subgroup of G.

4. (2 points) True or False? $GL(5)$ is a topological group.

L20

2/27 The Fundamental Group.

Recall:

- $\Omega_{x_0} X$ is the set of all loops in X based at x_0
- $\underset{F}{\simeq}$ rel $\{0,1\}$ is an equivalence relation identifying homotopic loops (at x_0)
- $\langle \alpha \rangle \in \pi_1(X, x_0) :=$ the equivalence classes

Def: (loop concatenation) Binary operation $\pi_1(X,x_0) \times \pi_1(X,x_0) \to \pi_1(X,x_0)$ by $(\langle\alpha\rangle, \langle\beta\rangle) \mapsto \langle\alpha\rangle \cdot \langle\beta\rangle$ is well-defined.

Pf: Let $\langle\alpha\rangle = \langle\alpha'\rangle$, $\langle\beta\rangle = \langle\beta'\rangle$

Then $\alpha \underset{F}{\simeq} \alpha'$ and $\beta \underset{G}{\simeq} \beta'$ (rel $\{0,1\}$)

Now define $H : I \times I \to X$ by $(s,t) \mapsto \begin{cases} F(2s,t) & 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$

H is continuous [pasting lemma]

Note that $\alpha \cdot \beta \underset{H}{\simeq} \alpha' \cdot \beta'$ rel $\{0,1\}$

So $\langle \alpha \cdot \beta \rangle = \langle \alpha' \cdot \beta' \rangle$

Thm: (The Fundamental Group) $(\pi_1(X,x_0), \cdot)$ is a group.

Pf: Note we already have well-defined binary operation.

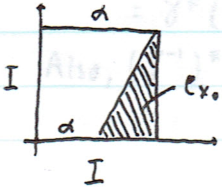
① Associativity ✓ [L19, last prop.]

② Identity: Define $e_{x_0} : I \to X$ by $s \mapsto x_0$ (i.e. a constant function)

Let $\langle\alpha\rangle \in \pi_1(X, x_0)$

[goal: show $\langle\alpha\rangle \cdot \langle e_{x_0}\rangle = \langle\alpha\rangle = \langle e_{x_0}\rangle \cdot \langle\alpha\rangle$, so we want to find a homotopy F s.t. $\alpha \cdot e_{x_0} \underset{F}{\simeq} \alpha$]

Define $F : I \times I \to X$ by $(s,t) = \begin{cases} \alpha\left(\frac{s}{\frac{1}{2}+\frac{t}{2}}\right) & 0 \le s \le \frac{1}{2} + \frac{t}{2} \\ x_0 & \frac{1}{2} + \frac{t}{2} \le s \le 1 \end{cases}$



(Note: image of $F(\cdot, t)$ is independent of t)

→

Thm: (The Fundamental Group)

Pf: ③ Inverse: Let $\langle \alpha \rangle \in \pi_1(X, x_0)$

recall $\alpha^{-1}: I \to X$ by $s \mapsto \alpha(1-s)$

So $\langle \alpha^{-1} \rangle$ is the inverse for $\langle \alpha \rangle$

Pf: If $\langle \alpha \rangle = \langle \alpha' \rangle$, then $\alpha \overset{\simeq}{F} \alpha'$ rel $\{0,1\}$.

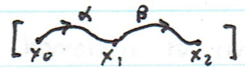
Define $G(s,t) = (1-s, t)$.

Then $\alpha^{-1} \overset{\simeq}{G} (\alpha')^{-1}$ rel $\{0,1\}$.

To show $\langle \alpha \rangle \cdot \langle \alpha^{-1} \rangle = \langle e_{x_0} \rangle = \langle \alpha^{-1} \rangle \cdot \langle \alpha \rangle$, consider

$$F: I \times I \to X \text{ by } (s,t) \mapsto \begin{cases} \alpha(1-t)2s & 0 \le s \le \frac{1}{2} \\ \alpha((1-t)(2-2s)) & \frac{1}{2} \le s \le 1 \end{cases}$$ ▯.

Note: $\langle \alpha \rangle \cdot \langle \beta \rangle := \langle \alpha \cdot \beta \rangle$



∘ α, β don't have to be loops: if they're not, it's called a groupoid

Thm: X is path-connected. $x_0, x_1 \in X$. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Pf: Since X path-connected, $\exists \gamma: I \to X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Define $\gamma^*: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\langle \alpha \rangle \mapsto \langle (\gamma^{-1} \cdot \alpha) \cdot \gamma \rangle$

$= \langle \gamma^{-1} \cdot (\alpha \cdot \gamma) \rangle$ [assoc.]

Then γ^* is well defined [LTR]

recall: $\phi(g_1 g_2) = \phi g_1 \phi g_2$ inverse preserved.

γ^* is a group homomorphism.

Pf: $\gamma^*(\langle \alpha \rangle \cdot \langle \beta \rangle) = \gamma^*(\langle \alpha \cdot \beta \rangle) = \langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle$

$= \langle \gamma^{-1} \cdot \alpha \rangle \cdot \langle \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \rangle \cdot \langle e_{x_0} \rangle \cdot \langle \beta \cdot \gamma \rangle$

$= \langle \gamma^{-1} \cdot \alpha \rangle \cdot \langle \gamma \cdot \gamma^{-1} \rangle \cdot \langle \beta \cdot \gamma \rangle$ [e_{x_0} groupoid identity]

$= \langle \gamma^{-1} \cdot \alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle \cdot \langle \gamma^{-1} \cdot \beta \cdot \gamma \rangle$

$= \gamma^*(\langle \alpha \rangle) \cdot \gamma^*(\langle \beta \rangle)$

Also, $(\gamma^{-1})^*: \pi_1(X, x_1) \to \pi_1(X, x_0)$ is the inverse [LTR] ▯.

Problems for Lesson 20: The Fundamental Group

February 27, 2017

Problem (2) will be graded.

(1)

- Let $f : X \to Y$ be a continuous function, $x_0 \in X$ and $y_0 = f(x_0)$. Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$. Prove that f_* is a well-defined group homomorphism. (We say f_* is the group homomorphism induced from the continuous function f.)

- if $id : X \to X$ is the identify function and $x_0 \in X$, then $id_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity function on $\pi_1(X, x_0)$.

- If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, $x_0 \in X$, $y_0 = f(x_0)$ and $z_0 = g(y_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_* : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ satisfy $(g \circ f)_* = g_* \circ f_*$.

Remark. Rendered in modern language, the above says that the fundamental group construction is a **functor** from the category of based topological spaces and continuous functions to the category of groups and group homomorphisms. This is just one example of such functors. We will see another after the spring vacation. **Algebraic topology** is a subject in mathematics, which studies such functors from some topological category to some algebraic category (and hence the name).

Hint: The textbook has a discussion. But that's incomplete. You need to supply all the details.

(2) Prove that if $f : X \to Y$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a group isomorphism.

Remark. This means if you can show that the fundamental groups of two spaces are not isomorphic, then the two spaces are not homeomorphic. This is the power of algebraic topology. But the power is limited: there is no reason to believe that the converse statement is also true. You will see that the spaces in the shapes of the english letters A, O and P have the same fundamental group, but it's not difficult to see that they are pairwise non-homeomorphic.

3/1 Computations I: Path/Homotopy-Lifting Lemmas

Recall: If X is path-connected, then π_1 doesn't depend on x_0 [L20]

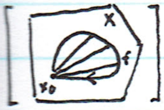
∘ so we can write $\pi_1(X)$

Def: X is simply-connected if:

① X is path-connected

② $\pi_1(X) \cong \{0\}$

∘ e.g: $\mathbb{R}^n$, D^n, any convex subset of $\mathbb{R}^n$



Pf: Use straight-line homotopy $F: I\times I \to X$ by $F(s,t) = (1-t)f(s) + t x_0$ where $\langle f\rangle \in \pi_1(X, x_0)$

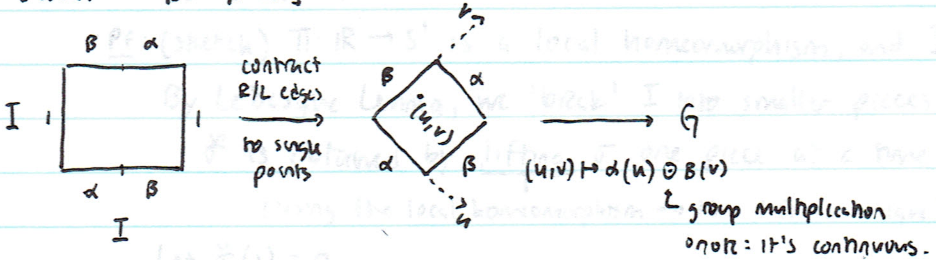
∘ note: so any path based at x_0 is homotopic to constant path ($\langle f\rangle = \langle e_{x_0}\rangle$)

Thm: Let G be a path-connected topological group. Then $\pi_1(G)$ is abelian.

∘ note: not all $\pi_1(X)$ are abelian, e.g. $\pi_1(K)$ is not abelian (the Klein bottle)

Pf: Let $\langle\alpha\rangle, \langle\beta\rangle \in \pi_1(G, 1)$ [1 is identity]

[Goal: $\alpha\cdot\beta \simeq \beta\cdot\alpha$]



So we have $F: I\times I \to G$, so $\alpha\cdot\beta \overset{F}{\simeq} \beta\cdot\alpha$ rel $\{0,1\}$

Pf: LTR (Eckman-Hilton argument in homework)

Ex: $(S^1, 1)$ is a topological group → so $\pi_1(S^1)$ must be abelian

∘ note: $\pi_1(S^1) \cong \mathbb{Z}$ (see next page)

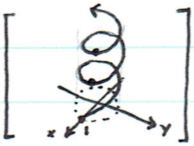
Cor: S^1 and D^2 are not homeomorphic

Pf: use L20, P2 □

Prop: $\pi_1(S^1) \cong \mathbb{Z}$

Pf: (sketch) Key idea: $\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{i2\pi s}$ ("Slinky projection" : )

Note: $\mathbb{Z}$ gets mapped to 1 (on the unit circle)



For each $n \in \mathbb{Z}$, define path $\gamma_n : [0,1] \to \mathbb{R}$ by $s \mapsto ns$

Now define $\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ by $n \mapsto \langle \pi \circ \gamma_n \rangle$

So n is sent to the homotopy class of paths that wrap around S^1 $|n|$ times, counterclockwise if $n \geq 0$, clockwise if $n < 0$.

Fact: Φ is an isomorphism.

Pf: ① Φ is a homomorphism

Pf: $\Phi(m+n) = \langle \pi \circ \gamma_{m+n} \rangle$

$= \langle \pi \circ (\gamma_m \cdot \alpha) \rangle$

$= \langle (\pi \circ \gamma_m) \cdot (\pi \circ \alpha) \rangle$

$= \langle \pi \circ \gamma_m \rangle \cdot \langle \pi \circ \gamma_n \rangle = \Phi(m) \cdot \Phi(n)$

② Φ is onto

Pf: Let $\langle \sigma \rangle \in \pi_1(S^1, 1)$

$$\left[\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ I \arrow[ur, dotted, "\hat{\sigma}"] \arrow[r, "\sigma"'] & S^1 \end{tikzcd}\right]$$

Path-Lifting Lemma: Then $\exists$ path $\hat{\sigma} : I \to \mathbb{R}$ s.t. $\hat{\sigma}(0) = 0 \in \mathbb{R}$ and $\pi \circ \hat{\sigma} = \sigma$

Pf: (sketch) $\pi : \mathbb{R} \to S^1$ is a local homeomorphism, and I is compact. By Lebesgue Lemma, we 'break' I into smaller pieces, and $\hat{\sigma}$ is obtained by lifting σ one piece at a time.

(using the local homeomorphism → this lift is unique)

Let $\hat{\sigma}(1) = n$.

Then straight-line homotopy F shows $\hat{\sigma} \underset{F}{\simeq} \gamma_n$

Thus $\Phi(n) = \langle \pi \circ \gamma_n \rangle = \langle \pi \circ \hat{\sigma} \rangle = \langle r \rangle$

③ Φ is 1:1

Pf: Suppose $\Phi(n) = \langle \pi_* \circ \gamma_n \rangle = \langle e_1 \rangle$ ← constant loop at 1. [wts: $n = 0$]

$$\left[\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ I \times I \arrow[ur, dashed, "\hat{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}\right]$$

Homotopy-Lifting Lemma: Then $\exists$ unique $\hat{F} : I \times I \to \mathbb{R}$ s.t. $\hat{F}(0,t) = 0 \quad t \in I$, $\pi \circ \hat{F} = F$

Pf: similar to p-l lemma

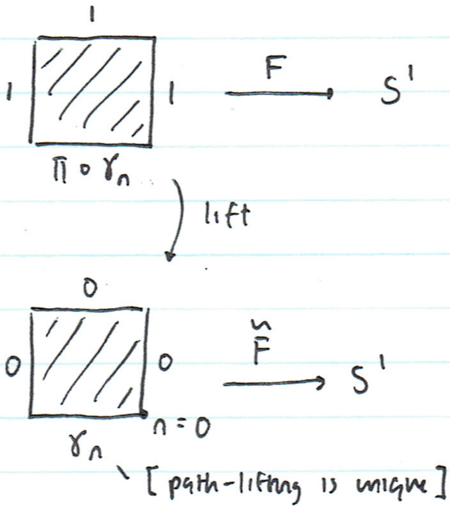
So $\pi \circ \gamma_n \underset{\hat{F}}{\simeq} e_1$

→

Prop: $\pi_1(S^1) \cong \mathbb{Z}$

<u>Pf</u>: Φ isomorphism

<u>Pf</u>: ③ (cont'd) Consider the below:



Problems for Lesson 21: Computations: the Path/Homotopy-Lifting Lemmas

March 1, 2017

Problem (2) will be graded.

(1) In class, we sketched the proof of $\pi_1(S^1) \cong \mathbb{Z}$. Read Page 97-98 for details. Even though this is more like a project than a homework problem, I still encourage you to recover the proof on your own. It uses several important ideas.

(2) In class, we proved that π_1 of a connected topological group is abelian. Though the proof is interesting, it is rather ad hoc. This problem invites you to prove the statement again using the Eckmann-Hilton argument, which can also be applied to many other problems in mathematics.

Let α, β, γ and δ be four loops in the topological group $(G, \odot, 1)$ based at 1, where $\odot$ is the group operation. Then it is a fact (which you can check on your own) that

$$(\alpha \odot \beta) \cdot (\gamma \odot \delta) = (\alpha \cdot \gamma) \odot (\beta \cdot \delta),$$

where $\alpha \odot \beta$ and similar terms denote pointwise group multilication, i.e., $\alpha \odot \beta : I \to G$ is defined by $(\alpha \odot \beta)(s) = \alpha(s) \odot \beta(s)$.

Prove that $\pi_1(G, 1)$ is abelian. Notice that if e_1 denotes the constant path at $1 \in G$, then $\langle \alpha \rangle \cdot \langle \beta \rangle = \langle e_1 \odot \alpha \rangle \cdot \langle \beta \odot e_1 \rangle$ and similar identity also hold.

(3) Who is this mathematician (picture on the left)? What is the title of his Ph.D. thesis in its original language?

(4) Who is this mathematician (picture on the right: second person from right, taken from the movie The Imitation Game 2014)? What is the title of his Ph.D. thesis in its original language? *Hint:* He broke codes with Alan Turing during WWII.

3/2 Computations II: Unions and Products

Recall: $\pi_1(S^1) \cong \mathbb{Z}$ [L21]

- what about $\pi_1(S^n)$? For $n \geq 2$, $\pi_1(S^n) \cong \{0\}$...

[note: this thm generalizes to Seifert-vanKampen theorem]

Thm: Let $X = U \cup V$ where U, V simply-connected and $U \cap V$ is path-connected and nonempty. Then X is also simply-connected.

Pf: ① X is path-connected.

Pf: U, V simply-connected, so U, V path-connected. Since $U \cap V \neq \emptyset$, then $X = U \cup V$ is path-connected [L13, P3]

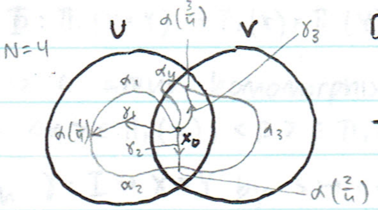
② $\pi_1(X) \cong \{0\}$

may not be open... need to rewrite.

Pf: Let $\langle \alpha \rangle \in \pi_1(X, x_0)$ where $x_0 \in U \cap V$ [$U \cap V \neq \emptyset$]

Since $\alpha^{-1}U, \alpha^{-1}V$ cover $[0,1]$, which is compact, by Lebesgue's Lemma $\exists N \in \mathbb{N}$ s.t. each $[\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1..N$ is either in $\alpha^{-1}U$ or in $\alpha^{-1}V$

So $\alpha([\frac{i-1}{N}, \frac{i}{N}])$ is either in U or V.



Define $\alpha_i : I \to X$ by
$s \mapsto \alpha(\frac{i-1}{N} + \frac{1}{N}s)$, $i = 1..N$

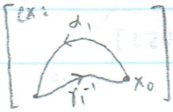
Then for each $i = 1..N-1$, we have three possible cases.

Case 1: $\alpha(\frac{i}{N}) \in U$. Since U is path-connected, connect $x_0 \in U \cap V \subseteq U$ to $\alpha(\frac{i}{N})$ by a path γ_i in U.

Case 2: $\alpha(\frac{i}{N}) \in V$. Similar to case 1, use γ_i in V.

Case 3: $\alpha(\frac{i}{N}) \in U \cap V$. Since $U \cap V$ is path-connected... use γ_i in $U \cap V$.

Note that $\langle \alpha \rangle = \langle \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_N \rangle = \langle \alpha_1 \rangle \cdot \ldots \cdot \langle \alpha_N \rangle$



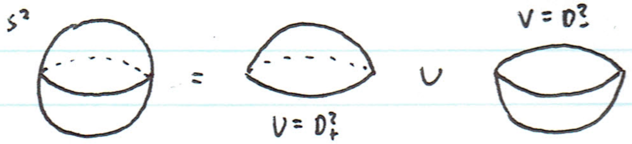
$= \langle \alpha_1 \rangle \cdot \langle e_{\alpha(\frac{1}{N})} \rangle \cdot \langle \alpha_2 \rangle \cdot \ldots \cdot \langle e_{\alpha(\frac{N-1}{N})} \rangle \cdot \langle \alpha_N \rangle$

$= \langle \alpha_1 \rangle \cdot \langle \gamma_1^{-1} \cdot \gamma_1 \rangle \cdot \langle \alpha_2 \rangle \cdot \ldots \cdot \langle \gamma_{N-1}^{-1} \gamma_{N-1} \rangle \cdot \langle \alpha_N \rangle$

$= \langle \alpha_1 \cdot \gamma_1^{-1} \rangle \cdot \langle \gamma_1 \cdot \alpha_2 \cdot \gamma_2^{-1} \rangle \cdot \ldots \cdot \langle \gamma_{N-1} \cdot \alpha_N \rangle$

$= \langle e_{x_0} \rangle \cdot \ldots \cdot \langle e_{x_0} \rangle = \langle e_{x_0} \rangle$ [$\pi_1(U), \pi_2(V) \cong \{0\}$] $\square$

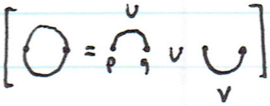
Ex: $\pi_1(S^2) \cong \{0\}$



- $U \cap V = S^1$, which is path-connected
- note: $D^2_\pm \cong D^2$, which is convex $\Rightarrow \pi_1(D^2) \cong \{0\}$

⟶ in general, $\pi_1(S^n) \cong \{0\}$, $S^n = D^n_+ \cup D^n_-$

- doesn't work for S^1, since $U \cap V = S^0 = \{p, q\}$, which is not path-connected.



$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$

Thm: If X, Y path-connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Pf: Let $x_0 \in X$, $y_0 \in Y$.

Let $p_1 : X \times Y \to X$ by $(x,y) \mapsto x$, $p_2 : X \times Y \to Y$ by $(x,y) \mapsto y$.

p_1, p_2 are continuous. [LTR].

Then let $(p_1)_* : \pi_1(X \times Y) \to \pi_1(X)$ by $\langle \alpha \rangle \mapsto \langle p_1 \circ \alpha \rangle$, $(p_2)_*$ similar.

Then $(p_1)_*, (p_2)_*$ are group homomorphisms [hw]

So $\Phi : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $\langle \alpha \rangle \mapsto ((p_1)_*(\langle \alpha \rangle), (p_2)_*(\langle \alpha \rangle))$ is a group homomorphism. (① ✓)

Let $\langle \alpha \rangle \in \pi_1(X)$, $\langle \beta \rangle \in \pi_1(Y)$.

Then $\gamma : I \to X \times Y$ by $s \mapsto \langle \alpha(s), \beta(s) \rangle$ is a loop in $X \times Y$ based at (x_0, y_0).

Clearly, $\Phi(\langle \gamma \rangle) = (\langle \alpha \rangle, \langle \beta \rangle)$. (② onto ✓)

Let $\langle \gamma \rangle \in \pi_1(X \times Y)$, and $\Phi(\langle \gamma \rangle) = (\langle e_{x_0} \rangle, \langle e_{y_0} \rangle)$

Then $p_1 \circ \gamma \underset{F}{\simeq} e_{x_0}$, $p_2 \circ \gamma \underset{G}{\simeq} e_{y_0}$.

[kernel trivial].

So $\gamma \underset{H}{\simeq} (e_{x_0}, e_{y_0})$ by $H(s,t) = (F(s,t), G(s,t))$. (③ 1:1 ✓)] □

Ex: $\pi_1(\underset{\text{torus.}}{T^2}) \underset{\uparrow \, [L20,\, P2]}{\cong} \pi_1(S^1 \times S^1) \underset{\uparrow \, [\text{thm}]}{\cong} \pi_1(S^1) \times \pi_1(S^1)$, i.e. $\mathbb{Z} \times \mathbb{Z}$

Problems for Lesson 22: Computations: Unions and Products

March 2, 2017

Problem (1) will be graded.

(1)
- Prove again the following theorem: Let $X = U \cup V$ where U and V are simply-connected and $U \cap V$ is nonempty and path-connected. Furthermore, we assume that U and V are open. Then X is simply-connected.
- Let X be obtained by gluing disjoint spaces $S^2 \times S^3$ and $S^3 \times S^2$ at a single point. Compute $\pi_1(X)$. State all theorems you used.

(2) Let X be a path-connected space and $x_0, x_1 \in X$. Prove that every pair of paths γ_1 and γ_2 from x_0 to x_1 induce the same isomorphism on the fundamental groups $\pi_1(X, x_0) \to \pi_1(X, x_1)$ if and only if $\pi_1(X, x_0)$ is abelian. (This problem serves as a review of concepts from Lesson 20.)

Math 455 Topology, Spring 2017
Exam 1
March 3

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) If $f : X \to Y$ is continuous and X is compact, then Y is also compact.

   (b) (2 points) S^2 is both connected and path-connected.

   (c) (2 points) $\mathbb{R}P^1$ is homeomorphic to S^1.

   (d) (2 points) If $f : X \to Y$ is a homeomorphism and Y is Hausdorff, then X is also Hausdorff.

   (e) (2 points) If C_i, $i \in I$ are compact subsets of X, then $\cup_{i\in I} C_i$ is also compact.

2. (10 points)

   (a) (5 points) Use definition to prove that the subspace $\{0\} \cup \{1/n \big| n = 1, 2, 3, \cdots\}$ of $\mathbb{R}$ is compact.

   (b) (5 points) Prove that the one-point compactification Y of a Hausdorff space X indeed is compact.

3. (10 points)

   (a) (7 points) Prove that if X is path-connected, then it is connected.

   (b) (3 points) Give an example of a space which is connected but not path connected.

4. (10 points) Prove that $[0,1]/\{0,1\}$ is homeomorphic to S^1. State precisely the theorem(s) you use.

5. (10 points)

   (a) (5 points) Prove that under matrix multiplication, $GL(2)$, the space of all 2 by 2 invertible real matrices, is a topological group.

   (b) (5 points) Prove that $(\mathbb{R}, +, 0)$ and $(\mathbb{R}_{>0}, \cdot, 1)$ are isomorphic as topological groups. Assume you have proved that they are topological groups. You only need to prove the isomorphic part.

MATH 455 Quiz #6 **Name:**____________________

Let the function $\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ be defined by $\Phi(n) = \langle \pi \circ \gamma_n \rangle$ where $\pi : \mathbb{R} \to S^1$ is the projection function given by $\pi(s) = e^{i2\pi s}$ and $\gamma_n : [0,1] \to \mathbb{R}, s \longmapsto ns$ is the uniform-speed path joining 0 and n in $\mathbb{R}$. We proved (sketched the proof) that Φ is actually an isomorphism.

1. (2 points) In which of the following steps is the Homotopy-Lifting Lemma used?
   (a) Φ is a group homomorphism.
   (b) Φ is onto.
   (c) Φ is one-to-one.

2. (2 points) In which (could be more than one) of the following steps is the Path-Lifting Lemma used?
   (a) Φ is a group homomorphism.
   (b) Φ is onto.
   (c) Φ is one-to-one.

3. (2 points) True or False? S^n is simply-connected for all $n = 0, 1, 2, 3, 4, 5, \cdots$.

4. (2 points) True or False? The fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

5. (2 points) True or False? (It's a fact that the fundamental group of the Klein bottle is not abelian.) Then it must be true that the Klein bottle is NOT a topological group.

3/6 "A for Amherst," Deformation Retraction, Homotopy Equivalence, and Contractibility

Recall: notions of equivalence

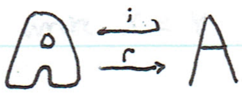
- e.g. two spaces are considered "the same" if $\exists$ a homeomorphism b/w them

↳ in $\mathbb{R}^2$:

: $f \circ g = \mathrm{id}_{\mathbb{A}}$

$g \circ f = \mathrm{id}_{\mathbb{A}}$

↳ in $\mathbb{R}^2$:



: $r \circ i = \mathrm{id}_A$

$i \circ r \boxed{\underset{F}{\simeq}} \mathrm{id}_{\mathbb{A}}$ [$F: \mathbb{A} \times I \to \mathbb{A}$ by $(x,t) \mapsto (1-t)\,\mathrm{id}(x) + t\underbrace{b_x}_{\text{base point.}}$]

- easy to see map from A into $\mathbb{A}$:

- in other direction: break $\mathbb{A}$ into disjoint closed intervals & contract.
  - i.e. collapse each interval onto its base point

Def: Let $A \subseteq X$. Let $i: A \hookrightarrow X$ be the inclusion.

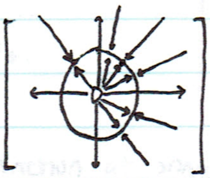
① $r: X \to A$ is called a retraction if $r \circ i = \mathrm{id}_A$

② If also $\exists$ a homotopy rel A, $F: X \times I \to X$, s.t. $\mathrm{id}_X \underset{F}{\simeq} i \circ r$ rel A, then $r: X \to A$ is a deformation retraction

- note: F is also called a deformation retraction
- note: A is a deformation retract, and X deformation retracts to A
- note: $r = F(\cdot, 1)$

Ex: $X = \mathbb{R}^2 \setminus \{(0,0)\}$, $A = S^1$. Then X deformation retracts to A.

<u>Pf</u>: Consider $F: X \times I \to A$ by $(x,t) \mapsto tx + (1-t)\frac{x}{|x|}$



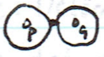
Note this is rel A (i.e. if $x \in A$, F indep. of t)

<u>Pf</u>: Let $x \in A = S^1$

Then $|x| = 1$. So $tx + (1-t)\frac{x}{1} = x$.

Consider $r: X \to A$ by $x \mapsto F(x,1)$

Ex: $X = \mathbb{R}^2 \setminus \{p,q\}$ $(p \neq q)$. Many deformation retractions, e.g:



"8"

"θ"

(see textbook)

So: Homeomorphism $X \underset{g}{\overset{f}{\rightleftarrows}} Y$: $f \circ g = id_Y$, $g \circ f = id_X$

Def. Retraction $X \underset{i}{\overset{r}{\rightleftarrows}} Y$: $f \circ g = id_Y$, $i \circ r \simeq id_X$

Homotopy Equivalence $X \underset{g}{\overset{f}{\rightleftarrows}} Y$: $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$

Def: X and Y are homotopy equivalent if $\exists$ maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $f \circ g \simeq id_X$, $g \circ f \simeq id_Y$

- note: f, g are called homotopy equivalences
- note: we write $X \simeq Y$ (careful! re-use of notation)

Thm: Being homotopy equivalent ($\simeq$) is an equivalence relation

Pf: LTR (on homework) □

Cor: If $X \cong Y$, then $X \simeq Y$

Pf: Since X, Y homeomorphic, we know $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $f \circ g = id_Y$, $g \circ f = id_X$

So define $F(y,t) = y$, $G(x,t) = x$.

Then $f \circ g \underset{F}{\simeq} id_Y$, and $g \circ f \underset{G}{\simeq} id_X$.

Cor: If X def. retracts to A, then $X \simeq A$

Pf: LTR (similar)

Ex: So since $\mathbb{R}^2 \setminus \{p,q\} \simeq 8$, $\mathbb{R}^2 \setminus \{p,q\} \simeq \theta \longrightarrow 8 \simeq \theta$

Def: X is contractible if $X \simeq *$, a one-point space.

Note: how do we prove that a given X is contractible?

- $X \simeq *$ means that $X \underset{g}{\overset{f}{\rightleftarrows}} *$
  - $f \circ g \simeq id_*$: since $*$ is one point, we know $f \circ g = id_*$
  - $g \circ f \simeq id_X$: but $g \circ f$ must be constant map $c_{g(*)}$

$\longrightarrow$ So it suffices to prove that $id_X \simeq$ a constant function at some pt in X.

Ex: $\mathbb{R}, \mathbb{R}^2, [0,1]$ are all contractible

Note: Deformation retraction to a point and contractibility are not the same

- in def. retract, point can't move during F
- e.g. comb space (textbook pg. 108)

Problems for Lesson 23: A for Amherst, Deformation Retraction, Homotopy Equivalence and Contractibility

March 6, 2017

Problem (2) will be graded.

(1) Prove that being homotopy equivalent $\simeq$ is an equivalence relation. It's in the textbook.

(2) (a) Prove that the two-dimensional **thick** letter A on the Euclidean plane deformation retracts to each of the one-dimensional letters D, O, P, Q, and R, respectively. Write all the details for the letter D by imitating what we did for A in class. For the others, only draw the deformation retraction pictures.

(b) Prove that A, D, O, P, Q, and R are pairwise homotopy equivalent.

Comment: So you probably don't want to think of the alphabet as homotopy equivalence classes when you compose an English essay.)

(3) Check the details for the comb space example. The comb space is contractible but it does not deformation retracts to the point on the upper left corner. (Figure 5.10 in the textbook)

(4) Figure 5.12 in the textbook shows the famous "**house with two rooms**". Imagine how you can get it from a solid cylinder by deformation retraction. On the other hand, the solid cylinder obviously deformation retracts to a point. Since deformation retractions are homotopy equivalences. By the transitivity of $\simeq$, we know this avant-garde room is homotopy equivalent to a point and thus is contractible by definition.

(5) Assuming Problem 27, try to see if you can show that the "**dunce hat**" (Figure 5.11 in the textbook) is contractible.

L24

3/8 The Effect of Homotopy on f_x (and thus on Homotopy Equivalence Spaces)

Recall: (hw) if $f: X \to Y$ by $x_0 \mapsto y_0$, then $f_*: \pi_1(\overset{\text{loop at } x_0}{X, x_0}) \to \pi_1(\overset{\text{loop at } y_0}{Y, y_0})$ is a well-defined group homomorphism [L20, P1] $\langle \alpha \rangle \longmapsto \langle f \circ \alpha \rangle$

Recall: if $\gamma: I \to Y$ s.t. $\gamma(0) = y_0$, $\gamma(1) = y_1$, then $\gamma_*: \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is a group isomorphism $\langle \alpha \rangle \longmapsto \langle (\gamma^{-1} \cdot \alpha) \cdot \gamma \rangle$

Thm: If $f \underset{F}{\simeq} g: X \to Y$, $x_0 \in X$, then $g_* = \gamma_* \circ f_*$ where $\gamma: I \to Y$ by $s \mapsto F(x_0, s)$

- note: $\gamma(0) = F(x_0, 0) = f(x_0)$, $\gamma(1) = F(x_0, 1) = g(x_0)$
- diagram:

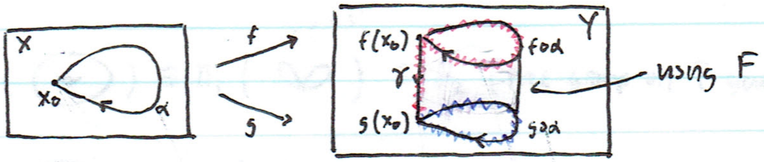
$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[d, "\gamma_*"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[r, "g_*"'] \arrow[ur, phantom, shift right=4, "\circlearrowright"] & \pi_1(Y, g(x_0))
\end{tikzcd}$$

commutative

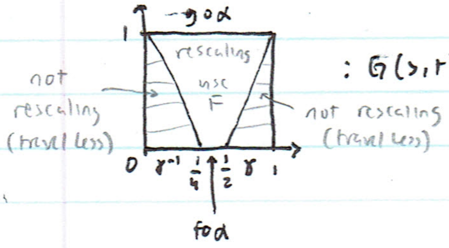
- i.e. f_* and g_* are the 'same' (if you don't care about isomorphism γ_*)

Pf: Given $\langle \alpha \rangle \in \pi_1(X, x_0)$

[wts: $(\gamma_* \circ f_*)(\langle \alpha \rangle) = g_*(\langle \alpha \rangle)$, i.e. $\langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle = \langle g \circ \alpha \rangle$]



↳ we see the homotopy: "moving down" the cylinder as $t \to 1$



$$: G(s,t) = \begin{cases} \gamma(1-4s) & 0 \le s \le \frac{1-t}{4} \\ F\left(\alpha\left(\frac{s - 1-t/4}{1+t/2 - 1-t/4}, t\right), t\right) & \frac{1-t}{4} \le s \le \frac{1+t}{2} \\ \gamma(2s-1) & \frac{1+t}{2} \le s \le 1 \end{cases}$$

□

homotopy equivalent

Thm: If $X \simeq Y$, then $\pi_1(X, x_0) \overset{\xrightarrow{f}}{\cong} \pi_1(Y, f(x_0))$

Pf: $X \simeq Y$ means $\exists\ X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$

Consider $g \circ f \simeq id_X$.

Then $\exists$ path γ s.t. $\gamma_* \circ (g \circ f)_* = (id_X)_*$ [prev. thm]

So $\gamma_* \circ g_* \circ f_* = id_{\pi_1(X, x_0)}$ [L20, P1.3 & P1.2]

So f_* is 1:1. [recall: $A \underset{b}{\overset{a}{\rightleftarrows}} B \Rightarrow b \circ a = id_A \Rightarrow a$ 1:1, b onto]

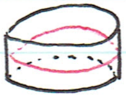
Consider $f \circ g \simeq id_Y$.

Then $\exists$ path δ s.t. $\delta_* \circ (id_Y)_* = (f \circ g)_*$

So $\delta_* = f_* \circ g_*$ [$(id_Y)_* = id_{\pi_1(Y)}$]

So $id_{\pi_1(Y)} = f_* \circ g_* \circ (\delta_*)^{-1}$, so f_* is onto.

Then f_* is a group isomorphism [$f: X \to Y$, $f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$]

Ex: $C^2 =$  $M^2 =$  : both deformation retract to belt circle

↳ i.e. $\simeq$

- i.e. $\pi_1(C^2) \cong \pi_1(S^1) \cong \pi_1(M^2)$
- note: solid torus (think: radial retraction of discs) is also the same.

$S^1 \vee S^1$

Ex: $\pi_1($  $) \cong \pi_1($  $) \cong F_2$ (free group on 2 generators)

Ex: $\pi_1($  $) \cong \pi_1($  $) \cong F_3$

$S^1 \vee S^1 \vee S^1$

Problems for Lesson 24: The Effect of Homotopy on f_* and thus on Homotopy Equivalent Spaces

March 8, 2017

Problem (2) will be graded.

(1) Consider the following examples of a circle A embedded in the space X.
   (a) $X = \mathbb{R}^2 \backslash \{(0,0)\}$, A is the embedded standard circle S^1;
   (b) X is a circular cylinder, A is one of its boundary circles;
   (c) $X = T^2$, $A = \{(x,x) \in S^1 \times S^1\}$;
   (d) X is a Möbius trip, A is the boundary circle;
   In each case, describe the generators of the fundamental groups for A and X. Also describe the image in $\pi_1(X)$ of a generator of $\pi_1(A)$ under the homomorphism induced from the inclusion.

(2) Now you are ready to rigorously prove the following intuitively obvious fact. Let $\alpha : I \to X$ and $\beta : I \to X$ be two paths in the space $X = \mathbb{R}^2 \backslash \{(0,0)\}$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \text{ and } \beta(s) = (\cos(\pi s), -\sin(\pi s)).$$

Prove that $\alpha \not\simeq \beta$ rel $\{0,1\}$. Justify all your claims. ((1a) is helpful.)

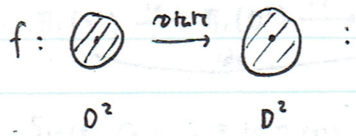
(3) Compute the fundamental group of a torus with one point removed (one puncture).

(4) Compute the fundamental group of the torus with two disjoint closed discs removed.

(5) Compute the fundamental group of the real projective plane with one puncture.

3/9 The Brouwer Fixed-Point Theorem

Story: Brouwer swirls a cup of coffee...



: notice that point in middle stays in same place
↳ i.e. $f(x)=x$, so x is a fixed point.
∘ Fact: ∀ continuous $f: D^2 \to D^2$, ∃ at least one fixed point

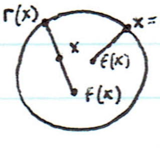
Thm: (Brouwer Fixed-Point Theorem) Let D^n be n-dimensional closed unit disc in $\mathbb{R}^n$, $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. Then ∀ maps $f: D^n \to D^n$, ∃ $x \in D^n$ s.t. $f(x) = x$
(continuous, standard topology)

∘ note: so f has a fixed point x.
∘ note: so D^n has the fixed point property (since any $f: D^n \to D^n$ has a fixed point)

Pf: ($n=1$) Follows from Intermediate Value Theorem [L12, P4]
($n \geq 3$) Use homology, or Lefschetz fixed-point theorem [in future]
We consider the case when $n=2$.

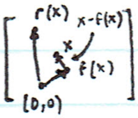
[Outline: (pf by contradiction) [many methods of proof]
∘ Assume ∃ $f: D^2 \to D^2$ s.t. f has no fixed point.
↳ Then D^2 retracts to S^1 ※ [shown by looking at π_1, f_*]]

Pf: Suppose ∃ $f: D^2 \to D^2$ s.t. ∀ $x \in D^2$, $f(x) \neq x$
Now, we can construct a retraction $r: D^2 \to S^1$



∀ $x \in D^2$, $f(x) \neq x$, so we can draw a ray from $f(x)$ thru x
: ∘ note: if $x \in \partial D^2 = S^1$, then $x = r(x)$
If $i: S^1 \to D^2$ by $x \mapsto x$, then $r \circ i = \mathrm{id}_{S^1}$ [so r is a retraction]

Also, r is continuous.



Pf: Let $r(x) = f(x) + t(x - f(x))$, where $t > 0$
We know $|r(x)| = 1$, so $|r(x)|^2 = r(x) \cdot r(x) = 1$
So $|x-f(x)|^2 t^2 + 2f(x) \cdot (x - f(x))t + |f(x)|^2 - 1 = 0 \to$ solve for t.
So $r(x) = f(x) + t(x - f(x))$

$$\nwarrow \frac{-f(x) \cdot (x - f(x)) + \sqrt{[f(x) \cdot (x - f(x))]^2 + 1 - |f(x)|^2}}{|x - f(x)|^2}$$

So r is continuous □

Thm: [Brouwer Fixed-Point Theorem]

Pf: (cont'd) Now, consider the maps:

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] \arrow[rr, bend right, "id_{S^1}"'] & D^2 \arrow[r, "r"] & S^1 \end{tikzcd}$$

top.

Apply π_1 functor

alg.

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "id_*"'] & \pi_1(D^2) \arrow[r, "r_*"] & \pi_1(S^1) \end{tikzcd}$$

$[r_* \circ i_* = (r \circ i)_* = id_* = id_{\pi_1(S^1)}]$

Since $r_* \circ i_* = id_{\pi_1(S^1)}$, then r_* is onto

But $r_* : \pi_1(D^2) \to \pi_1(S^1)$

$$\begin{tikzcd} \pi_1(D^2) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \pi_1(S^1) \arrow[d, phantom, sloped, "\cong"] \\ \{0\} & \mathbb{Z} \end{tikzcd}$$

⨳ $\square$

no surjection from singleton to $\mathbb{Z}$

Prop: Fixed-point property is a topological invariant.

Pf: Assume X has FPP, and $X \cong Y$ (i.e. $X \underset{h}{\overset{g}{\rightleftarrows}} Y$) [wts: Y has FPP]

Let $f: Y \to Y$ be an arbitrary map (f is a self-map)

Consider: $X \xrightarrow{g} Y \xrightarrow{f} Y \xrightarrow{h} X$

Since X has FPP, $\exists\, x \in X$ s.t. $(h \circ f \circ g)(x) = x$.

So $f(g(x)) = h^{-1}(x) = g(x)$ $\square$

Ex: $D^2 \cong$ "pac-man" ◕, so any f: ◕ $\to$ ◕ has a fixed-point

Note: FPP is not a homotopy invariant.

Problems for Lesson 25: The Brouwer Fixed-Point Theorem

March 9, 2017

Problem (1) will be graded.

(1) (a) State and prove the Brouwer Fixed-Point Theorem again.
    (b) Construct a continuous map from the open unit disk on $\mathbb{R}^2$ to itself such that it does not have a fixed point.

(2) If every continuous map from X to itself (called a self-map of X) has a fixed point and Y is homotopy equivalent to X, is it true that every self-map of Y also has a fixed point?

(3) (a) Prove that if A is a retract of X, then if every self-map of X has a fixed point, then every self-map of Y also has a fixed-point.
    (b) Prove that every self map of the House With Two Rooms has a fixed point.

(4) Whose Ph.D. defense ceremony is this? Hint: He also established the mathematical philosophy of *intuitionism*.

onto homomorphism

3/10 Application of "Retraction induces epimorphism"

(recall: used this in L25)

$a_0 \in A$

Thm: If $r : X \to A$ is a retraction, then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is an epimorphism (i.e. onto homomorphism)

<u>Pf</u>: Let $i : A \hookrightarrow X$ denote the inclusion map.

By def. of retraction, then $r \circ i = id_A$

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend right, "idA"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

So $(r \circ i)_* = (id_A)_*$, i.e. $r_* \circ i_* = id_{\pi_1(A, a_0)}$

So r_* is onto, and we know it is a homomorphism □

"open neighborhood of x"

i.e. $\exists\ f : \mathbb{R}^2 \xrightarrow{\cong} U$, $0 = (0,0) \mapsto x$

Def: A space S is a surface if: ① S is Hausdorff

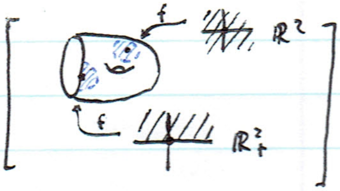
② $\forall x \in S$, $\exists$ open set $U \ni x$ in S s.t. U is: a) homeomorphic to $\mathbb{R}^2$, or

b) homeomorphic to $\mathbb{R}^2_+$

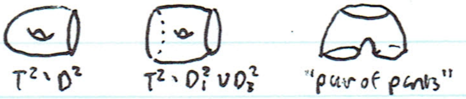
Ex: S^2, T^2, K^2 (the Klein bottle)

Ex: $\mathbb{R}P^2$, $T^2 \# T^2$

Ex: cylinder, M^2 (note: these have boundaries)



Ex:



Def: S surface. The interior of S consists of the points in S which have neighborhoods homeomorphic to $\mathbb{R}^2$

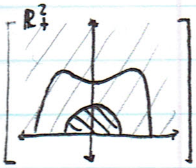
Def: S surface. The boundary of S consists of the points in S which have neighborhoods homeomorphic to $\mathbb{R}^2_+$

Thm: The interior and the boundary of a surface S are disjoint

<u>Pf</u>: Suppose $\text{int}\, S \cap \partial S \neq \emptyset$. Let $x \in \text{int}\, S \cap \partial S$

So $\exists$ open sets $U, V \ni x$ in S and homeomorphisms $f : \mathbb{R}^2 \to U$, $0 \mapsto x$, $g : \mathbb{R}^2_+ \to V$, $0 \mapsto x$.

Then $U \cap V$ is open in V [def. subspace topology]



So $g^{-1}(U \cap V)$ is open in $\mathbb{R}^2_+$ [g homeomorphism]

and $g(D_1) \subseteq U \cap V \subseteq U$

Since $0 \in g^{-1}(U \cap V)$ [$x \in U \cap V$], $\exists$ basic open set D_1 centered at 0 in $g^{-1}(U \cap V)$

Then $g(D')$ is open in $U \cap V$ and $U \cap V$ is open in U, i.e. $g(D')$ is open in U.

So $f^{-1} \circ g(D_1)$ is open in $\mathbb{R}^2$ and $0 \in f^{-1} \circ g(D_1)$

→

Thm: The interior and the boundary of a surface S are disjoint.

Pf: (cont'd) By def. top. on $\mathbb{R}^2$, $\exists$ open disk $B_\varepsilon(0) \subseteq f^{-1}\circ g(D_1)$

Let $D_2 = \overline{B_{\varepsilon/2}(0)}$

circle at 0 with radius $\frac{\varepsilon}{2}$

Now consider $r: f^{-1}\circ g(D_1)\setminus\{0\} \to C_{\varepsilon/2}(0)$ by $r(x) = \frac{\varepsilon}{2}\cdot\frac{x}{|x|}$

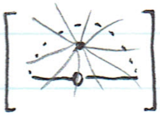
Note that r is continuous

In fact, r is a retraction because $r\circ i = id_{C_{\varepsilon/2}(0)}$

So, we have

$$r_*: \underset{\underset{\textstyle \pi_1(D_1\setminus\{0\})}{\cong}}{\pi_1(f^{-1}\circ g(D_1)\setminus\{0\})} \to \underset{\underset{\textstyle \mathbb{Z}}{\cong}}{\pi_1(C_{\varepsilon/2}(0))}$$

since $f^{-1}\circ g(D_1)\setminus\{0\} \cong D_1\setminus\{0\}$



But consider $D_1\setminus\{0\}$

Clearly, it deformation retracts to the point $(0,1)$

Problems for Lesson 26: Another Application of "Retraction Induces Epimorphism": Surfaces, their Interiors and their Boundaries

March 10, 2017

Problem (2) will be graded.

(1) Now you can show that the Möbius strip and the cylinder are not homeomorphic, even though both deformation retracts to S^1 (and thus are homotopy equivalent).

Hint: This is Corollary 5.25 of Theorem 5.24, which follows from the Theorem we proved in class.

(2) We proved in Lesson 12 that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic using the fact that connectedness is a topological invariant. We also mentioned in Problem (5) of Lesson 12 the failure of this method in showing that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$. Now you are ready to prove this fact: use an argument in the proof of the last theorem today to show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$. Hint: understand the proof the theorem in class thoroughly.

Also enjoy the Spring Vocation thoroughly! =)

MATH 455 Quiz #7 **Name:**____________________

1. (2 points) True or False? $\mathbb{R}^2\backslash\{(0,0)\}$ deformation retracts to S^1.

2. (2 points) True or False? There is a retraction from S^1 to the point $(1,0) \in S^1$.

3. (2 points) True or False? The space $\mathbb{R}^2\backslash\{p,q\}$, where $p \neq q \in \mathbb{R}^2$, and the space in the shape of the number 8 have isomorphic fundamental groups.

4. (2 points) True or False? Any continuous function from $[0,1]$ to $[0,1]$ has a fixed point.

5. (2 points) True or False? Möbius strip and cylinder are not homeomorphic.

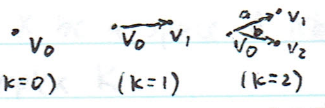
3/20 Simplex, Complex, Polyhedron, and Triangulations (LEGO)

Motivation: category of topological spaces & cont. fns is too vast to study

- luckily, almost all of the spaces/maps we've seen either:
  - ① Can be cut into simpler pieces, or
  - ② Have the homotopy type of spaces with ①
- advantages of combinatorial nature: easier proofs, compute π_1 for more spaces [L30, L31], classify surfaces [L32-36], define/compute homology [L37-45], applications of homology: maps [L44-46], knots [L47-49]

Def: k+1 points are in <u>general position</u> if $v_1 - v_0, \dots, v_k - v_0$ are lin. indep. (in $\mathbb{R}^n$)

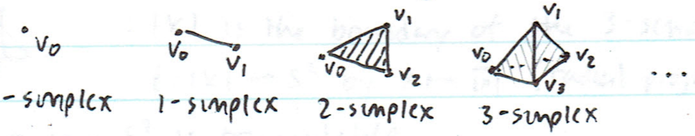
- e.g.:



… [intuition: not "coplanar" points]

Def: The smallest convex set containing $\overbrace{v_0, \dots, v_k}^{\text{(in general position)}}$ is a <u>k-simplex</u>.

- e.g.:



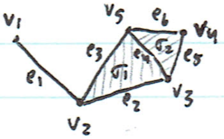
- note: the <u>boundaries</u> of simplices are also simplices
  - i.e. if $\{v_{i_0}, \dots, v_{i_m}\} \subseteq \{v_0, \dots, v_k\}$, then the smallest convex set containing $\{v_{i_0}, \dots, v_{i_m}\}$ is an m-simplex.
    - this is called a <u>face</u> of $A := \text{conv}\{v_0, \dots, v_k\}$
  - ex: 2-simplex has 7 faces (3 0-faces, 3 1-faces, 1 2-face)

Def: Let K be a <u>finite</u> collection of simplices in $\mathbb{R}^n$. Then K is a (simplicial) <u>complex</u> if:

① If simplex $A \in K$, then each face of A is also in K

② If $A_1, A_2 \in K$ and $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap A_2$ is a face of A_1 and A_2

- note: the <u>dimension</u> of K is the largest k s.t. a k-simplex is in K

Ex: 

$K = \{v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, e_5, e_6, \sigma_1, \sigma_2\}$

: $\dim K = 2$

note: this is a [illegible] complex

Ex: 

: this is not a complex

- notice that

Def: Given complex K in $\mathbb{R}^n$, let $|K|$ be the union of all the underlying simplices in K. Equip $|K|$ with the subspace topology of $\mathbb{R}^n$. Then $|K|$ is the polyhedron on K.

- note: K is just a collection of names.

Def: Let X be a space. A triangulation of X consists of:

① a complex K

② a homeomorphism $f: |K| \to X$

Ex: Let $X = S^2$ $\quad K = \{v_0, v_1, v_2, v_3, e_1, e_2, e_3, e_4, e_5, e_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

: $|K|$ is the boundary of the 3-simplex

$f: |K| \to S^2$ by $x \mapsto \frac{x}{|x|}$ [radial projection]

- then we say S^2 is triangulable

Def: Two complexes K and L are isomorphic if $\exists$ a bijection ϕ between their vertices s.t. $v_1, \dots, v_k \in K$ form the vertices of a simplex in K $\Leftrightarrow$ $\phi(v_1), \dots, \phi(v_k) \in L$ form the vertices of a simplex in L

- i.e. K and L have the same combinatorial structure

Problems for Lesson 27: Simplex, Complex, Polyhedron and Triangulation

March 20, 2017

Problem (2) will be graded.

(1) A standard n-simplex Δ^n is defined by

$$\Delta^n := \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \big| x_i \geq 0 \text{ for all } i, x_1 + x_2 + \cdots + x_{n+1} = 1\}.$$

Another set of standard simplices Γ^n is defined by $\Gamma^0 = \mathbb{R}^0$ and for $n \geq 1$,

$$\Gamma^n := \{(y_1, y_2, \cdots, y_n) \in \mathbb{R}^n \big| 0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq 1\}.$$

   (a) Sketch Δ^n for $n = 0, 1, 2$ and Γ^n for $n = 0, 1, 2, 3$ and then compare them.

   (b) For each n, find a linear transformation $T : \mathbb{R}^{n+1} \to \mathbb{R}^n$ mapping Δ^n homeomorphically onto Γ^n.

(2) (a) Find a triangulation of S^1. By definition, this means (1) find a simplicial complex K in some $\mathbb{R}^n$ and (2) find a homeomorphism $f : |K| \to S^1$. What is the minimal number of 0-simplices that you need?

   (b) Find another triangulation of S^1 for which the simplicial complex is **NOT isomorphic** to the one you used in (a).

   (c) Find a triangulation of the cylinder

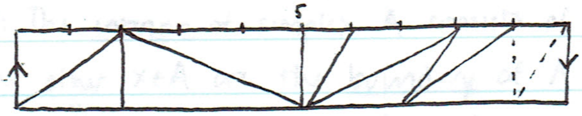
$$X := \{(x, y, z) \in \mathbb{R}^3 \big| x^2 + y^2 = 1, -1 \leq z \leq 1\}.$$

(3) Read through Lemma 6.3 on Page 124. It's a good opportunity to review most of the point-set topology we learned from Chapter II to IV.

(4) The idea of triangulation is to cut a space into "curved simplices", which are simpler building blocks. Simplicies are simple geometric objects, but they are not the only ones. For example, (hyper-)cubes are also simple enough. Try to build a similar theory to what we did today but using cubes: define cubes in all dimensions, define cubical complex, define its polyhedron and then define "*cubiculation*". In the older days, cubes were used in topology as often as simplices. In Jean-Pierre Serre's work leading to his 1954 Fields Medal, you can find homology defined using cubes instead of simplices.

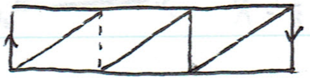
3/22 Origami, Cones, and Barycentric Subdivision

In HW: found triangulation of cylinder
- what about the Möbius strip?

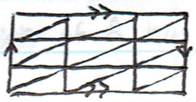


when folded (| = mountain, ⁝ = valley), : all faces/edges are straight, i.e. the Möbius strip is triangulable

- fact: 6 triangles suffice in $\mathbb{R}^n$ for $n \geq 4$

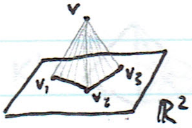


: consider also the triangulation of the Klein bottle (n large) :

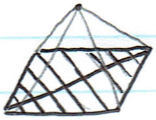


Def: Let K be a complex in $\mathbb{R}^n$. A cone on K, CK, is a complex in $\mathbb{R}^{n+1}$ defined as follows: Pick a point v in $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$. Then $A \in CK$ if: ① $A = v$, or ② $A \in K$, or ③ A is the simplex having $v_1, \ldots, v_k, v$ as vertices where $v_1, \ldots, v_k$ are the vertices for some $B \in K$

Ex:



Ex:



(filled-in pyramid)

Note: If K has N elements, then CK has $2N+1$ elements

Note: If we change the location of v, then we get an isomorphic complex
- so up to isomorphism, CK is unique ("CK is the cone on K")

Ex: Triangulation of $\mathbb{RP}^2$ =  ∪ 

Pf: Let K be the complex triangulating M.

subset of K that is also a complex

Let L be the subcomplex of K triangulating the boundary circle

Then $K \cup CL$ gives a triangulation for $\mathbb{RP}^2$

[$v \in \mathbb{R}^4 \setminus \mathbb{R}^3$  (10) $\cong D^2$]

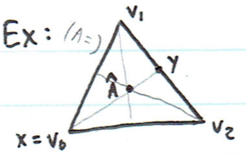
Fact: Given a simplex A with vertices $v_0, \dots, v_k$, then $\forall x \in A$ there are unique $\lambda_i \geq 0$ $(i = 0, \dots, k)$ s.t. $\lambda_0 + \dots + \lambda_k = 1$, and $x = \lambda_0 v_0 + \dots + \lambda_k v_k$.

- these λ_i are the barycentric coordinates of x.

Def: The interior of simplex A consists of those $x \in A$ for which all $\lambda_i > 0$.

- the other $x \in A$ are the boundary of A

Ex:



$x = 1 \cdot v_0 + 0 \cdot v_1 + 0 \cdot v_2$

$: y = 0 \cdot v_0 + \frac{1}{2} \cdot v_1 + \frac{1}{2} \cdot v_2$

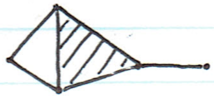
$\hat{A} = \frac{1}{3} \cdot v_0 + \frac{1}{3} \cdot v_1 + \frac{1}{3} \cdot v_2$ [in general: barycenter of A has $\lambda_i = \frac{1}{k+1}$]

Def: Let K be a complex. Then the barycentric subdivision of K, K', is formed by connecting the barycenters of K's faces in some way.

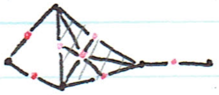
- the N^{th} barycentric subdivision $K^N := (K^{N-1})'$

Ex:

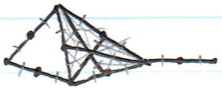
$K =$



$K' =$



$K^2 =$



Fact: The diameter of each simplex in K^n is $\leq \left(\frac{k}{k+1}\right)^n$ $(k = \dim K)$

- so as $n \to \infty$, diameter $\to 0$

Problems for Lesson 28: Origami, Cones and Barycentric Subdivision

March 22, 2017

Problem (2) will be graded.

(1) In class, the first complex in $\mathbb{R}^3$ triangulating the Möbius strip we saw has 10 triangles. Can you find a triangulation of the Möbius strip using fewer triangles? (For example, the second we saw in class has fewer triangles.) *Hint: Start from the diagram with six triangles we drew in class and see how many more you need.*

(2) (a) Let K be the complex in $\mathbb{R}^3$ consisting of the standard simplex Δ^2 from the homework of L27 and all its faces. Draw a picture illustrating the second barycentric subdivision K^2. How many simplicies in total are there in K^2?

  (b) How many triangles do you need for a triangulation of the torus T^2? (Answer is in the book!) Why can't you just use twelve? Draw a polyhedron $|K|$ for T^2 in $\mathbb{R}^3$.

(3) Read through Lemma 6.4 on Page 126.

3/23 Simplicial Approximation

This is a key concept moving forward:

- enables combinatorial computation of π_1
- shows homology is well-defined
- enables alternate proofs of previous results

Def: Let K, L be complexes. A map $s: |K| \to |L|$ is simplicial if s takes each simplex of K linearly onto a simplex of L.

- "linearly": $A \in K$, A spanned by vertices $v_0, \dots, v_k$. So $\forall x \in A$, $x = \sum_{i=0}^{k} \lambda_i v_i$ [L28].
  - ↳ then $s(x) = s(\Sigma \lambda_i v_i) = \Sigma \lambda_i s(v_i)$
- "onto": $s(v_i)$ must be vertices of L $(i = 0, \dots, k)$
  - note: $s(v_i)$ don't have to be distinct, i.e. not necessarily 1:1.

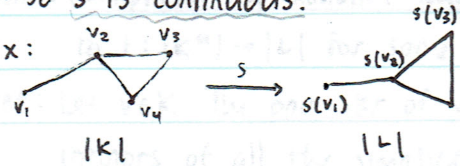
Note: s is determined by its effect on vertices of K (finitely many points!)

Note: s is continuous on each simplex (b/c it's linear)

- the restriction of S on each simplex matches perfectly → gluing lemma

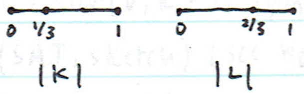
↳ so s is continuous.

Ex:



: s is onto for each simplex
- not over the whole complex

Ex:



: $f: I \to I$ by $x \mapsto x^2$ is not linear
- usually, maps are not linear...

⟶ BUT we can approximate a version of these maps.

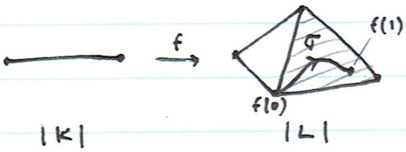
Def: Let $f: |K| \to |L|$ be a map. Then $s: |K| \to |L|$ is a simplicial approximation (of f) if:

① s is a simplicial map, and

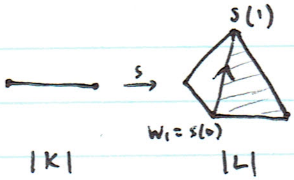
② $\forall x \in |K|$, $s(x)$ is in the unique simplex whose interior contains $f(x)$.

- i.e. $s(x)$ is in the carrier of $f(x)$.

Ex:



;



- s is simplicial
- carrier of $f(0)$ is w_1, of $f(t)$ is σ $(0<t\leq 1)$

→ s is simplicial approximation of f.

Thm: If s is a SA of f, then $s \underset{F}{\simeq} f$

Pf: use straight-line homotopy [a simplex is convex] [pts lie in same simplex] □.

Ex: Previous $x \mapsto x^2$ example has no SA (w/o barycentric subdivision)

Pf: Suppose s is a SA for $f: x \mapsto x^2$

Then $s(0) = f(0) = 0$, $s(1) = f(1) = 1$

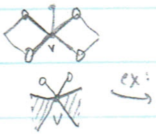
Also, $s(\frac{1}{3}) = \frac{2}{3}$ [it can't be 0 or 1]

So $s([0, \frac{1}{3}]) = [0, \frac{2}{3}]$, $s([\frac{1}{3}, 1]) = [\frac{2}{3}, 1]$

Note $f(\frac{1}{2}) = \frac{1}{4} \in [0, \frac{1}{3}] \Rightarrow s(\frac{1}{2}) \in [0, \frac{1}{3}]$ ※

- note: $f: |K^2| \to |L|$ by $x \mapsto x^2$ can be SA (by next thm)

Thm: (Simplicial Approximation Theorem SAT) There is a SA $s: |K^m| \to |L|$ to $f: |K^m| \to |L|$ for some m which is big enough.



Def: Let $v \in K$. The open star of v in K, $\operatorname{star}(v, K)$, is the union of the interiors of all the simplices in K having v as a vertex.

- note: $\operatorname{star}(v, K)$ is open in the subspace topology of K.

Pf: (SAT, sketch) (see book for details)

$\{\operatorname{star}(v, L) \mid v \text{ vertex of } L\}$ is an open cover of $|L|$

So $\{f^{-1}\operatorname{star}(v, L)\}$ is an open cover of $|K|$.

Since $|K|$ compact*, By Lebesgue's Lemma, $\exists N \in \mathbb{N}$ s.t. if $m \geq N$, then each $\operatorname{star}(u, K^m) \subseteq f^{-1}\operatorname{star}(v, L)$

* (union of compacts)

Recall: it suffices to define $s: |K^m| \to |L|$ on vertices.

Then s is a SA of f [properties of star] □

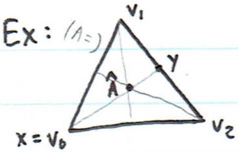
Fact: Given a simplex A with vertices $v_0, \dots, v_k$, then $\forall x \in A$ there are unique $\lambda_i \geq 0$ $(i = 0, \dots, k)$ s.t. $\lambda_0 + \dots + \lambda_k = 1$, and $x = \lambda_0 v_0 + \dots + \lambda_k v_k$.

- these λ_i are the barycentric coordinates of x.

Def: The interior of simplex A consists of those $x \in A$ for which all $\lambda_i > 0$.

- the other $x \in A$ are the boundary of A

Ex:



$x = 1 \cdot v_0 + 0 \cdot v_1 + 0 \cdot v_2$

: $y = 0 \cdot v_0 + \frac{1}{2} \cdot v_1 + \frac{1}{2} \cdot v_2$

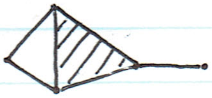
$\hat{A} = \frac{1}{3} \cdot v_0 + \frac{1}{3} \cdot v_1 + \frac{1}{3} \cdot v_2$ [in general: barycenter of A has $\lambda_i = \frac{1}{k+1}$]

Def: let K be a complex. Then the barycentric subdivision of K, K', is formed by connecting the barycenters of K's faces in some way.

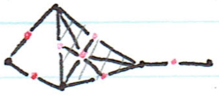
- the N^{th} barycentric subdivision $K^N := (K^{N-1})'$

Ex:

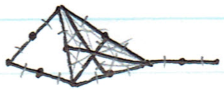
$K =$



$K' =$



$K^2 =$



Fact: The diameter of each simplex in K^n is $\leq \left(\frac{k}{k+1}\right)^n$ $(k = \dim K)$

- so as $n \to \infty$, diameter $\to 0$

Problems for Lesson 29: The Key Idea: Simplicial Approximation

March 23, 2017

Problem (3) will be graded.

(1) To prepare for tomorrow (Friday)'s lesson, read the Appendix (Page 241 – Page 243).

(2) Check the detail of the last example in class today: Prove that $f : |K^2| \to |L|$ can be simplicially approximated using Step 1 of the proof of the Simplicial Approximation Theorem.

(3) In Lesson 22, we proved that S^n is simply connected for $n \geq 2$. Now use the Simplicial Approximation Theorem to give a second proof.

Hint: if you can show that any map $\alpha : I \to S^n$ *can be deformed so that it misses at least one point on* S^n*, then stereographic projection tells you that* α *can actually be shrunk to a point.*

Comment: Do not assume that a map from an interval to S^n *is not surjective. This is not true. See Section 3 of Chapter 2 for the reason.*

(4) Use the Simplicial Approximation Theorem to prove that the set of homotopy classes of maps from one polyhedron to another is always countable. In particular, (with relative homotopy taken into consideration) it shows that if X has the homotopy type of a space which is triangulable, then $\pi_1(X)$ is a countable group. This puts a strong restriction on what kind of groups π_1 can be.

(5) There is a proof (by M. W. Hirsch) of the Brouwer fixed-point theorem using the Simplicial Approximation Theorem. You can look it up and read it.

3/25 Computing π_1: The Edge Group and its Convenient Presentation

Overview: Let X be triangulable. Let K be the complex.

- then $\pi_1(X) \cong \pi_1(|K|) \cong E(K,v) \cong G(K,L)$

Def: An edge loop based at v in K is a sequence of vertices $v_1, \dots, v_k$ s.t.:

① $v_0 = v_k = v$

② For each $i = 0, \dots, k-1$, either $v_i = v_{i+1}$ or v_i, v_{i+1} span an edge.

Consider an equivalence relation on the set of all edge loops based at v in K:

① $\cdots uvw \cdots \sim \cdots uw \cdots$ if uvw span a simplex $[{}_u\overset{v}{\Delta}_w \sim {}_u\overset{v}{\Delta}_w]$

② $uvu \sim u$

③ $uu \sim u$

$\longrightarrow$ Def: $E(K,v)$ is those equivalence classes

(loop concatenation)

Thm: $E(K,v)$ can be made into a group: $\{vv_1 \cdots v_k v\} \cdot \{vw_1 \cdots w_\ell v\} = \{v \cdots v_k v w_1 \cdots\}$

- identity: $\{v\}$
- inverse: $\{vv_1 \cdots v_k v\}^{-1} = \{vv_k \cdots v_1 v\}$

Thm: $\varphi : E(K,v) \to \pi_1(|K|, v)$ is a group isomorphism by $\{v \cdots v\} \mapsto \langle \alpha \rangle$

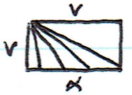
- $\alpha : I \to |K|$ by $0 \mapsto v$, $1 \mapsto v$, $\frac{i}{k+1} \mapsto v_i$ (rest is linear extension)

<u>Pf</u>: (sketch) Clearly, this is a group homomorphism

① Well-defined: equivalent sequences are sent to homotopic paths ✓

② Onto: Apply SAT to $\alpha : \overset{I=}{|L|} \to |K|$ where $L = \{0, 1, I\}$

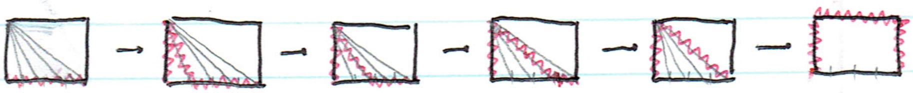
③ 1-1: Apply SAT to $F : |J| \to |K|$ where J =



$[\alpha \underset{F}{\overset{\text{n}}{\simeq}} e_v]$ null-homotopic loop

So there is $m \to L^m =$

We map this diagram on L^m to $|K|$:



□

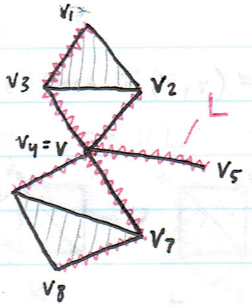
In K, the edges (1-simplices) form a graph. Let L be a tree (i.e. L has no loops) in K which is maximal (i.e. contains all vertices in K). Then L is a subcomplex.

Def: $G(K,L) = \Big\langle g_{ij} \;\Big|\;$ ① If v_i, v_j, v_k $\overset{(i<j<k)}{}$ span 2-simplex of K, then $g_{ij}g_{jk} = g_{ik}$
② If v_i, v_j span 1-simplex of L, then $g_{ij} = 1$ ← the identity

- note: b/c of ②, we don't write g_{ij} if v_i, v_j span 1-simplex of L

Thm: $G(K,L) \cong E(K,v)$

Ex:



$|K|$

: $G(K,L) = \langle g_{12}, g_{23}, g_{67}, g_{68} \mid$

- by ①: $g_{12}g_{23} = \underbrace{g_{13}} = 1 \Rightarrow g_{12} = g_{23}^{-1}$
- by ②: $\longrightarrow 1 \quad\quad 1$
- by ①: $g_{67}\overbrace{g_{78}} = g_{68} \Rightarrow g_{67} = g_{68}$

So $G(K,L) = \langle g_{12}, g_{23}, g_{68} \mid g_{12} = g_{23}^{-1} \rangle$

$\cong \langle g_{12}, g_{68} \rangle$

$\cong E(K,v) \cong \pi_1(|K|)$ $\begin{bmatrix} g_{12} \mapsto \{vv_3v_1v_2v\} \\ g_{68} \mapsto \{vv_6v_8v_7v\} \end{bmatrix}$

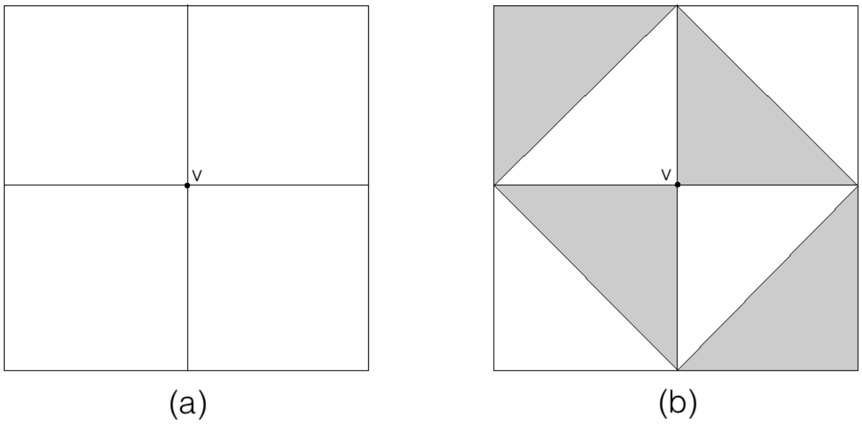
(get free group on 2 generators)

Problems for Lesson 30: Computing π_1: The Edge Group and its Convenient Presentation

March 24, 2017

Problem (1) will be graded.

(1) (a) Use $G(K, L)$ to compute the fundamental group of the left polyhedron $|K|$. Find all the elements of $E(K, v)$.
    (b) Use $G(K, L)$ to compute the fundamental group of the right polyhedron $|K|$. Find the simplest presentation of your group.



(2) From the computation of $G(K, L)$ and thus of $\pi_1(|K|)$, we see that it has nothing to do with simplices of dimension ≥ 3. Use this fact to give a third proof of the statement that $\pi_1(S^n)$ is trivial if $n \geq 2$.

Hint: $\pi_1(S^n) = \pi_1(D^n)$ if $n \geq 2$, where D^n is the solid ball whose boundary is S^n. You can also find the proof in the book.

(3) Read Theorem 6.10 and Theorem 6.12 in the textbook.

MATH 455 Quiz #8 **Name:**____________________

1. (2 points) True or False? $\mathbb{R}^2$ is not triangulable. (Recall that a simplicial complex has finitely many simplices.)

2. (2 points) True or False? $\mathbb{R}^2 \backslash \{(0,0)\}$ is homotopy equivalent to a space which is triangulable.

3. (2 points) True or False? For a complex K, $|K|$ is a topological space with its topology inherited from the Euclidean space it sits in.

4. (2 points) Let X be path-connected and $|K| \cong X$. Then $\pi_1(X) \cong E(K, v)$.

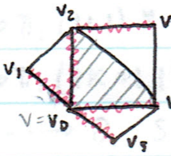
5. (2 points) What's the total number of faces of a 2-simplex?

3/28 Computing π_1: The Seifert van Kampen Theorem

Idea: ① Break space into 2 pieces

② π_1 of space is related to π_1s of pieces & their intersections

Ex: Let $|J|$ be:



we know $\pi_1(|J|, \overset{v_0}{v}) \cong E(J, v) \cong G(J, L)$

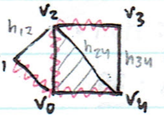
↳ so we find generators from $J \setminus L$

$\circ\ \langle g_{12}, g_{24}, g_{34}, g_{45} \mid g_{02} g_{24} = g_{04}$, but $g_{02} = g_{04} = 1 \rangle$

$\cong \langle g_{12}, g_{34}, g_{45} \rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

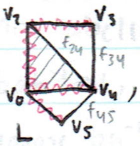
$\circ$ note that $E(J, v) \cong \langle \{v v_1 v_2 v\}, \{v v_2 v_3 v_4 v\}, \{v v_4 v_5 v\} \rangle$

$\circ$ write $J = K \cup L$:



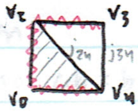
K

$\cup$



L

,



$K \cap L$

$\pi_1(|K|, v) \cong \langle h_{12}, h_{24}, h_{34} \mid h_{24} = 1 \rangle \cong \langle h_{12}, h_{34} \rangle$

$\pi_1(|L|, v) \cong \langle f_{34}, f_{45} \rangle$

$\pi_1(|K \cap L|, v) \cong \langle j_{34} \rangle \cong \mathbb{Z}$

$\circ$ Let $k : |K \cap L| \hookrightarrow |K|$, $\ell : |K \cap L| \hookrightarrow |L|$ be the inclusions.

↳ Then $k_*(j_{34}) = h_{34}$, $\ell_*(j_{34}) = f_{34}$

$\circ$ recall: $\pi_1(|J|, v) = \pi_1(|K \cup L|, v) \cong \langle g_{12}, g_{34}, g_{45} \rangle$

$\circ$ and $\pi_1(|K|, v) * \pi_1(|L|, v) \cong \langle h_{12}, h_{34} \rangle * \langle f_{34}, f_{45} \rangle = \langle h_{12}, h_{34}, f_{34}, f_{45} \rangle$

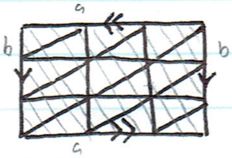
⟶ so: $\pi_1(|K \cup L|, v) \cong \pi_1(|K|, v) * \pi_1(|L|, v) / \sim$

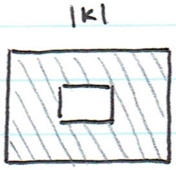
$\circ$ where $\sim$ is: for any generator z of $\pi_1(|K \cup L|, v)$, $k_*(z) = \ell_*(z)$

Thm: (Seifert-van Kampen Theorem) The fundamental group of $|K \cup L|$ based at v is obtained from free product $\pi_1(|K|, v) * \pi_1(|L|, v)$ by adding the relations $k_*(z) = \ell_*(z)$ for generator $z \in \pi_1(|K \cup L|, v)$.

Note: when we do computations, as long as the space is triangulable, we rarely care about what the triangulation is.

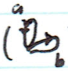
Ex: X = Klein Bottle

$X =$ 

$=$ 

$\cup$ 

: so $|K \cap L| =$  (boundary)

- note: a is a loop
- $\pi_1(|K|) \cong \pi_1($  $)$ [by radial contraction ] $\cong \pi_1($  $) \cong \langle a,b \rangle$

- $\pi_1(|L|) \cong \pi_1(\ast)$ [radial contraction to pt.] $\cong \{0\}$
- $\pi_1(|K \cap L|) \cong \langle z \rangle$ $(\cong \mathbb{Z})$

$\longrightarrow$ By SvK theorem, $\pi_1(\text{Klein Bottle}) \cong \pi_1(|K|) * \pi_1(|L|) / \sim$

$\cong \langle a,b \rangle * \{0\} / \sim \;\cong \langle a,b \rangle / \sim$

- where $k_*(z) = abab^{-1}$, $l_*(z) = \{0\}$ [follow radial retraction
  - ↳ so $\pi_1(\text{Klein Bottle}) \cong \langle a,b \mid abab^{-1} = 1 \rangle$
    - note: not abelian

      Pf: $abab^{-1} = 1$, so $ab = ba^{-1}$

      Suppose $ba^{-1} = ba$.

      Then $a^{-1} = a$, so $a^2 = 1$ ※ $\square$
  - ↳ so not a topological group.

Ex: $T^2 =$ 

$\cup$ 

- so $\pi_1(T^2) \cong \langle a,b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$
  - note: free abelian group

Problems for Lesson 31: Computing π_1: The Seifer-van Kampen Theorem

March 27, 2017

Problem (1) will be graded.

(1) (a) Use the Seifert-van Kampen Theorem to compute the fundamental group of $\mathbb{R}P^2$. (It's in the book. But try this on your own at least for the first one hour.)

  (b) Use the Seifert-van Kampen Theorem to compute the fundamental group of the two-holed torus $T^2 \# T^2$. (Cut it into two equal halves, or cut it into a disk and the rest. The first is easier than the second. Only write up one for your grader, but not both. Nonetheless, you should try both on your scratch paper. For the latter, consult Figure 7.20 in the book.)

(2) Read Theorem 6.13 in the textbook.

(3) Who are these two mathematicians? What are the titles of their Ph.D. dissertations in their original languages? Hint: The mathematician on the left wrote a classic textbook with his advisor *Lehrbuch der Topologie*. Both its original version and its translation to English are available in our library.

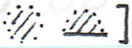
L32

3/29 Classification of Closed Surfaces: Statement of Result

Recall: A surface is a space X which is:

① Hausdorff

② $\forall x \in X$, $\exists$ open set U in X s.t. $x \in U$ and either $U \cong \mathbb{R}^2$ or $U \cong \mathbb{R}^2_+$

∘ note: we can replace $\mathbb{R}^2$, $\mathbb{R}^2_+$ with D^2, D^2_+ []

Def: A closed surface is a surface that is also:

③ connected [e.g.  not closed]

④ compact [e.g.  not closed]

⑤ no boundary [e.g.  not closed]

∘ "closed": closes back onto itself ("keep walking & end up where you started")

Note: A classification of objects (e.g. periodic table) must satisfy:

① Pairwise Different: no two items are the same

② Exhaustive: given arbitrary object, can find item which is the same.

Thm: (Classification Theorem for Closed Surfaces) Any closed surface is homeomorphic to one of the following, and no two of the following are homeomorphic:

① S^2

② S^2 with m handles added, $m = 1, 2, \ldots$

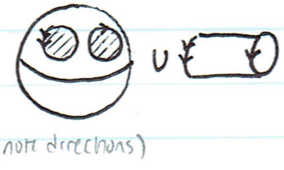
③ S^2 with n Möbius strips added, $n = 1, 2, \ldots$

∘ note: though closed surfaces are mostly characterized by local properties, this theorem characterizes the whole surface (!)

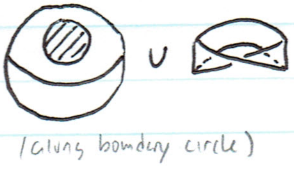
(proven fact)

∘ note: for higher-dimension ($n \geq 4$) surfaces, we can't classify them

∘ ② "adding handles": ③ "adding Möbius strip"



(note directions)



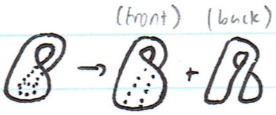
(along boundary circle)

Ex: S^2 w/ one handle added $\cong$ torus T^2

- 2 handles → 2-holed donut, 3 handles → pretzel surface

Ex: S^2 w/ one Möbius strip added $\cong \mathbb{R}P^2$ [note: $S^2 \setminus D^2 \cong \overline{D^2}$]

- 2 Möbius strips → Klein bottle
  - think: cut Klein bottle into 2 halves:



Thm: (Alternative Formulation of Class. Thm) Any surface is homeomorphic to either:

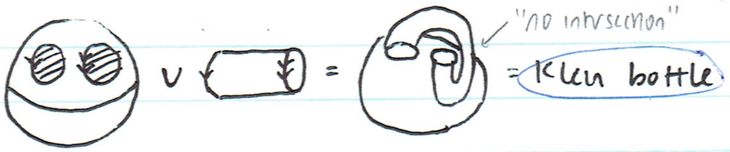
① S^2

② $mT^2 := T^2 \# T^2 \# \ldots \# T^2$

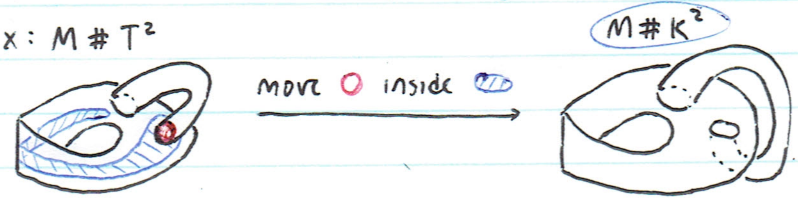
③ $n\mathbb{R}P^2 := \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ (and no two on the list are homeomorphic)

- note: # means removing open disc from both and gluing on boundary
  - ex:

- note: what if we don't flip orientations when "adding handles"?
  - ex:



- note: what if we combine ② and ③?
  - ex: $M \# T^2$



Problems for Lesson 32: The Classification of Closed Surfaces: Statement of Result

March 29, 2017

Problem (2) will be graded.

(1) Check the detail that mT^2 is homeomorphic to S^2 with m handles added and $n\mathbb{R}P^2$ is homeomorphic to S^2 with n discs removed and then n Möbius strips added.

(2) Explain your answers in detail. Draw pictures if necessary.

   (a) In the alternative statement of the classification theorem, what does the Klein bottle K^2 correspond to?

   (b) In the alternative statement of the classification theorem, what does $\mathbb{R}P^2\#T^2$ correspond to?

3/30 Prep. for Proof of Classification Theorem, I: Triangulation & Orientation

Thm: Any closed surface is triangulable $^{\text{by polyhedron of dim. 2}}$

Pf: (sketch, since pf is difficult)

Since surface S has no boundary, $\forall x \in S$ $\exists$ open set $U_x \ni x$ $\cong D^2$ $^{\text{open disc}}$

So we have an open cover $\{U_x \mid x \in S\}$ of S.

Since S is compact, $\exists$ finite subcover $U_1, \dots, U_N$

Then: ① If $U_i \subseteq U_j$, "throw away" U_i

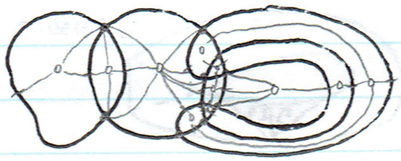
- note: we don't want annulus ( ^{U_j})

② Deform these U_i s.t. any $\underset{\text{two boundary circles}}{\underbrace{\text{two}}}$ of them intersect at only finitely many points

- note: this is difficult result (covering property preserved)

Now, use process similar to barycentric subdivision to triangulate S

- e.g.:

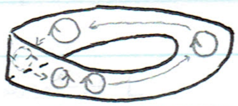


: Put point in "middle" of each region
Put vertex at each edge intersection
Put vertex at "midpoint" of each arc

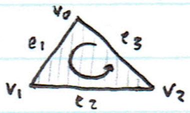
⟶ connecting these vertices produces complex K s.t. $|K| \cong S$ □

Def: If a surface S contains a Möbius strip, then S is non-orientable

- otherwise, S is orientable
- e.g. moving oriented circle around Möbius strip :



- we'd like to make this idea discrete
  - e.g. for a 2-simplex:

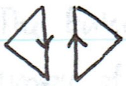


: $\{v_0v_1v_2, v_1v_2v_0, v_2v_0v_1\}$: this equivalence class is defined to be an orientation

↳ so $\{v_0v_2v_1, v_2v_1v_0, v_1v_0v_2\}$ is the other orientation

- def: also get induced orientation on the edges (v_0v_1 for e_1, etc.)

Def: The orientations on two adjacent triangles are compatible if the induced orientations on the common edge are opposite

◦ e.g.:  : these two triangles are compatible : 

Def: Polyhedron $|K|$ is orientable if all 2-simplicies can be oriented in a compatible manner

Prop: If surface S is orientable, then underlying $|K|$ is orientable

Pf: Start w/ arbitrary triangle, give it arbitrary orientation

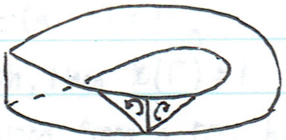
Iteratively, take all adjacent triangles an give them compatible orientations

Case 1: orientations compatible → ✓

Case 2: suppose there is a pair of triangles 

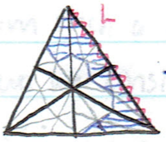
Then this is a Möbius strip contained in $|K|$, so in S ⨳

◦ idea:



Def: Let L be a 1-dimensional subcomplex of K (or of K'). Then a thickening of L in K is the 2-dimensional subcomplex of K^2 which consists of all simplices of K^2 which intersect $|L|$.

◦ ex:



→ ≡ is thickening of L

Problems for Lesson 33: Preparation for the proof, I: Triangulation and Orientation

March 30, 2017

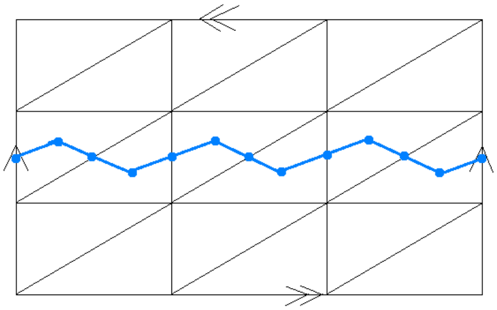
Problem (2) will be graded.

(1) Read the proofs of Lemma 7.3 and Lemma 7.4. (For 7.3, it's mainly because the union of two discs along an arc is again a disc.)

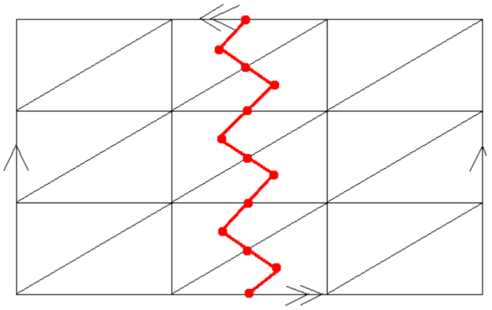
(2) Both complexes below triangulate the Klein bottle.

(a) Draw the thickening of the blue simple closed curve in Figure (a). Does that give a cylinder or a Möbius strip?

(b) Draw the thickening of the red simple closed curve in Figure (b). Does that give a cylinder or a Möbius strip?



(a)

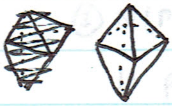


(b)

(3) https://math.osu.edu/about-us/history/tibor-radó

3/31 Prep. for Proof, II: Euler Characteristics

Def: Let L be a complex of dim n. The Euler characteristic of L, $\chi(L)$, $:= \sum_{i=0}^{n} (-1)^i \alpha_i$, where α_i is the number of i-simplices in L.

- ex:  $(|L| \cong S^2)$: $\chi(L) = \underbrace{4}_{\text{vertices}} - \underbrace{6}_{\text{edges}} + \underbrace{4}_{\text{faces}} = 2$
- ex:  $(|L| \cong S^2)$: $\chi(L) = 6 - 12 + 8 = 2$
- Fact: χ for surfaces does not depend on triangulation

  Pf: after we learn homology

Prop: If T is a tree, then $\chi(T) = 1$

Pf: The # of vertices in a tree is always one more than the # of edges.

Then $\chi(T) = \alpha_0 - (\alpha_0 - 1) = 1$ □

Prop: If Γ is a graph, then $\chi(\Gamma) \leq 1$.

Pf: A graph is obtained from a tree by adding edges.

Then $\chi(\Gamma) = \alpha_0 - \alpha_1$, where $\alpha_1 \geq \alpha_0 - 1 \rightarrow \chi(\Gamma) \leq 1$ □

$|K| \cong S$, where S is a closed surface

Thm: For any combinatorial surface K, $\chi(K) \leq 2$

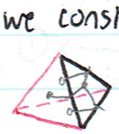
Pf: (idea: find tree T, graph Γ on $|K|$ s.t. $\chi(K) = \chi(T) + \chi(\Gamma) \leq 2$)

Given such a complex K, let T be a maximal tree in it.

Then we construct Γ, the dual graph to T, on $|K|$

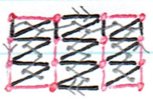
① : vertices are barycenters of the 2-simplices & barycenters of 1-simplices not in T

② : connect those vertices

- ex:  : T is one maximal tree

  Γ is the dual graph

In general, Γ is not a tree (in fact, ∀ surfaces but S^2)

- ex: Klein bottle:  : T: 

  Γ: 

→

Thm: For any combinatorial surface K, $\chi(K) \leq 2$

Pf: (cont'd) Γ indeed is a graph.

To see the proof, thicken both T and Γ in K^2.

Let the corresponding polyhedra for the two thickenings be $\overset{\text{"neighborhood"}}{N}(T)$ and $N(\Gamma)$.

Facts: ① $N(T) \cup N(\Gamma) = |K|$

② $\partial N(\Gamma) = \partial N(T) = N(\Gamma) \cap N(T) \cong S^1$

③ $N(\Gamma)$ deformation retracts to $|\Gamma|$, which means if $N(\Gamma)$ is path-connected, then so is $|\Gamma|$.

- ex: torus:

: $N(T)$ / $N(\Gamma)$: we can connect any arbitrary x, y in $N(\Gamma)$

So Γ is a graph.

So: $\chi(K) = v_K - e_K + f_K$ since $e_K = e_T + e_\Gamma$,

$\chi(T) = v_T - e_T$: then $\chi(K) = \underset{=1}{\chi(T)} + \underset{\leq 1}{\chi(\Gamma)} \leq 2$

$\chi(\Gamma) = v_\Gamma - e_\Gamma$

□

Thm: TFAE:

① Every $\overset{\text{no self-intersection}}{\overbrace{\text{simple}}}$ closed polygonal curve separates $|K|$ into 2 $\overset{\text{(path)}}{\text{components}}$

② $\chi(K) = 2$

③ $|K| \cong S^2$

Pf: (sketch)

$$\begin{tikzcd}
 & ② \arrow[dr] & \\
① \arrow[ur] & & ③ \arrow[ll, "\text{Jordan curve thm}"]
\end{tikzcd}$$

① → ② : Γ must be a tree. Otherwise, Γ has a loop, and then T can't be maximal.

② → ③ : $|K| = \underset{\overline{D^2}}{N(T)} \cup \underset{\overline{D^2}}{N(\Gamma)} \cong S^2$

on S^1 □

Problems for Lesson 34: Preparation for the proof, II: Euler Characteristics

March 31, 2017

Problem (2) will be graded.

(1) Check the details for the proof of the last theorem we stated in class.

(2) (a) Let K and L be arbitrary simplicial complexes which intersect in the subcomplex $K \cap L$. Prove that $\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$.

  (b) Let v, e and f be the numbers of vertices, edges and faces respectively of a simplicial complex K triangulating a closed surface. Find the number of vertices, edges and faces of the first barycentric subdivisioin K^1 of K. What is the relationship bewteen $\chi(K)$ and $\chi(K^1)$?

MATH 455 Quiz #9 **Name:**____________________

1. (2 points) True or False? The Euler characteristic of a connected tree is 1.

2. (2 points) True or False? The thickening of a simple closed curve in a combinatorial surface always gives a cylinder.

3. (2 points) True or False? $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, where T^2 is the torus.

4. (2 points) True or False? $\pi_1(K^2) \cong \langle a, b \big| aba^{-1}b = 1 \rangle$, where K^2 is the Klein bottle.

5. (2 points) What's the Euler characteristic of a simplicial complex which triangulates a circle?

4/3 Proof of Classification Theorem, I: Surgery tells us list is exhaustive

Idea: start from any combinatorial surface K ($K \cong S$, S closed surface)

~~Remarks~~ ∘ we will prove half of the classification theorem

Thm: $|K|$ is homeomorphic to either:

① S^2

② $mT^2 := T^2 \# \ldots \# T^2$

③ $n\mathbb{R}P^2 := \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$

Pf: If $|K|$ is S^2 → done (①)

Suppose $|K|$ is not S^2 [i.e. not homeomorphic to S^2]

Then ∃ simple closed polygonal curve on $|K|$ that does not separate $|K|$ into 2 components [L34, last thm (contrapositive)]

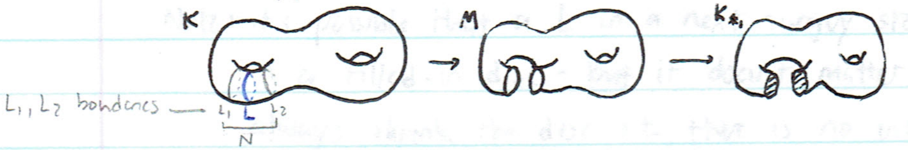
↳ pick such a curve: denote it by L [L is 1-dim complex]

Let N be the thickened 2-dim complex of L in K^2

Then $|N|$ is either a cylinder or a Möbius strip. [think: L33]

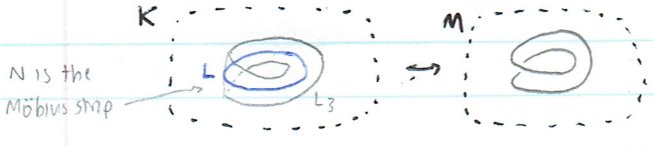
Let M be the subcomplex of K^2 s.t. $|M| = |K| \setminus$ interior of $|N|$

Case 1: $|N|$ is cylinder



: $K_{*_1} := M \cup CL_1 \cup CL_2$

Case 2: $|N|$ is Möbius strip



→ $K_{*_2} := M \cup CL_3$

∘ note: this process is called surgery

$$K \xrightarrow[\text{of non-sep. circle}]{\text{rem. thickening}} M \xrightarrow[\text{with disc(s)}]{\text{fill in hole(s)}} K_{*_i} \; ; \; i=1,2$$

K: original closed surface

M: surface with boundary

K_{*_i}: closed surface

→

Thm: Classification Theorem part 1

Pf: (cont'd) [L34 hw: $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$]

① $\chi(K_{*_1}) = \chi(M \cup CL_1 \cup CL_2) = \chi(M) + \chi(CL_1 \cup CL_2) - \chi(L_1 \cup L_2)$

$= \chi(M) + \chi(CL_1) + \chi(CL_2) - \chi(L_1) - \chi(L_2)$

- note: $\chi(S^1) = n - n = 0$ [△]
- note: $\chi(D^2) = (n+1) - 2n + n = 1$ []

$: 1+1-\chi(\text{line}) = 1+1-1 = 1$

↳ so 0, since $(1+1-1)-1 = 0$.

$02 \leftarrow (S^1) \rightarrow 02$

- note: $\chi(K) = \chi(M) + \chi(N) - \chi(M \cap N) = \chi(M) + \underbrace{\chi(N)} - [\chi(L_1) + \chi(L_2)]$

So $\chi(K_{*_1}) = \chi(M) + 2 = \chi(K) + 2$

② $\chi(K_{*_2}) = \chi(K) + 1$ [LTR]

So doing surgeries increases χ, i.e. whenever $\chi(K_{*_{i_1} *_{i_2} \cdots *_{i_n}}) < 2$, we can do another surgery until $\chi(K_{*_{i_1} \cdots *_{i_N}}) = 2$, i.e. $|K_{*_{i_1} \cdots *_{i_N}}| \cong S^2$ [L34]

Now, reverse the process

① If $|K|$ is orientable, $|K_{*_{i_1} * \cdots *_{i_N}}| \cong S^2$, then $|K|$ is NT^2

② If $|K|$ is nonorientable, $|K|$ is S^2 with i Möbius strips added and j handles added and N-i-j are cylinders added w/ reflection, then $|K| \cong i\mathbb{RP}^2 \# jT^2 \# (N-i-j)K^2$

$\cong \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$ [$\mathbb{RP}^2 \# T^2 = \mathbb{RP}^2 \# K^2$, K^2 is $2\mathbb{RP}^2$]

Note: it's possible that a L in a next surgery step could intersect in a filled-in disc - but it doesn't matter since we can always shrink the disc s.t. there is no intersection □

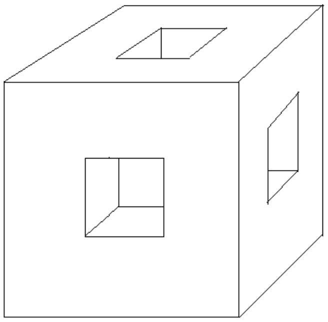
Problems for Lesson 35: Proof, I: Surgery tells us that the list is exhaustive.

April 3, 2017

Problem (2) will be graded.

(1) Using the results we proved in class about Euler characteristics ($\chi(K_{*_1}) = \chi(K) + 2$ and $\chi(K_{*_2}) = \chi(K) + 1$), find $\chi(mT^2)$ and $\chi(n\mathbb{R}P^2)$.

(2) Which two surfaces on the list in the classification theorem do the following two surfaces correspond to? (Think of a big cube as the union of 27 smaller cubes like those you saw in a rubik's cube. Remove seven cubes, six at the centers of the six faces and one at the very center of the big cube which you don't see from the outside. What you see in (a) is the surface of the remaining solid. (b) is a malicious cat used to be kept by Klein.)



(a)

(b)

4/5 Proof, II: π_1 tells us items on list are different

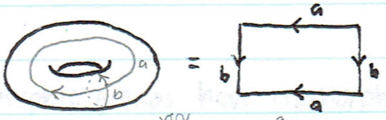
Recall: In L35, we proved that any closed surface is on the list:

① S^2 ⎡ ① S^2 ⎤
② mT^2 or ② $H(m)$: i.e. the list is exhaustive.
③ $n\mathbb{R}P^2$ ⎣ ③ $M(n)$ ⎦

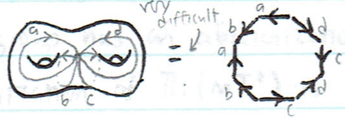
↳ we will prove: 1) the surfaces constructed in this way are well-defined
2) the surfaces are pairwise non-homeomorphic

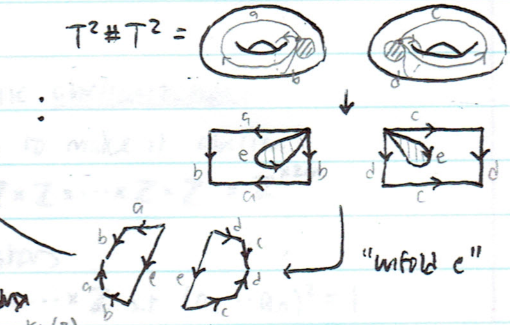
- sketch: polygonal models of mT^2, $n\mathbb{R}P^2$ → compute π_1 → abelianize π_1

Ex: T^2 =



$T^2 \# T^2$ =

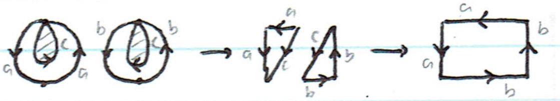


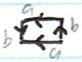


- so $\pi_1(T^2 \# T^2) = \pi_1($  $)$ $|K|$ $|L|$
$= \langle a,b,c,d \rangle * \{0\} / aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$ $k_*(2)$ $l_*(2)$
↳ $\pi_1(T^2 \# T^2) \cong \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

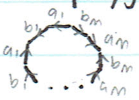
Ex: $\mathbb{R}P^2$ =

→ $\mathbb{R}P^2 \# \mathbb{R}P^2$:

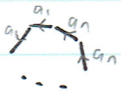


- note: homeomorphic to  (Klein bottle) [cut & reassemble]
- so $\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \langle a,b \mid a^2b^2 = 1 \rangle$

So using the polygonal models:

- $mT^2 =$ 

$\rightarrow \pi_1(mT^2) \cong \langle a_1, b_1, \ldots, a_m, b_m \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_m b_m a_m^{-1} b_m^{-1} = 1 \rangle$

- $n\mathbb{R}P^2 =$ 

$\rightarrow \pi_1(n\mathbb{R}P^2) \cong \langle a_1, \ldots, a_n \mid a_1^2 \cdots a_n^2 = 1 \rangle$

- recall that $\pi_1(S^2) \cong \{0\}$

Note: In general, two groups in generators & relations are extremely difficult to be told apart (if it's even possible)

- ex: $\langle a,b \mid abab^{-1} \rangle \cong \langle ab \mid a^2b^2 = 1 \rangle$ [Klein bottle] : look very different

Fact: Isomorphic groups have isomorphic abelianizations

- def: any group has an abelianization to make it abelian.

$\longrightarrow$ abelianization of $\pi_1(mT^2)$ is $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^{\times 2m}$

- free abelian group on $2m$ generators

$\longrightarrow$ abelianization of $\pi_1(n\mathbb{R}P^2)$ is $\mathbb{Z} \times \cdots \times \mathbb{Z}$ s.t. $(a_1 \cdots a_n)^2 = 1$

↳ use "change of basis":

$$\begin{array}{ccc} a_1 & & a_1 a_2 \cdots a_n \\ a_2 & \longrightarrow & a_2 \\ \vdots & & \vdots \\ a_n & & a_n \end{array}$$

$$\longrightarrow \mathbb{Z}/2\mathbb{Z} \times \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n-1}$$

So these groups are all different. □

Problems for Lesson 36: Proof, II: π_1 tells that the surfaces are different

April 5, 2017

Problem (1) will be graded.

(1) (a) Sketch the polygonal models for the two surfaces of L35 and also compute their fundamental groups.

  (b) Compute the fundamental group of the surface obtained by removing the interiors of r disjoint closed discs from mT^2.

  (c) Compute the fundamental group of the surface obtained by removing the interiors of r disjoint closed discs from $n\mathbb{R}P^2$.

  (d) Prove that the groups $\langle a, b \big| abab^{-1} = 1\rangle$ and $\langle a, b \big| a^2b^2 = 1\rangle$ are isomorphic. (Hint: Both are the fundamental groups of a well-known surface.)

(2) Classify connected and compact surfaces (not necessarily without boundary). This is outlined in the exercises on Page 170.

(3) One good problem to test your understanding of the tools we learned so far is Problem 33 on Page 171: identify the two surfaces having boundary with the standard ones. Hint: The number of caps you need to glue on to get a closed surface, Euler characteristics and orientability (in terms of orientation on simplicies) would be helpful.

Exam 2 Study Guide

Exam 2 will take place on **Friday, April 14th**, in our regular classroom **Seeley Mudd 207** during our regular class time from **11:00 A.M. to 11:50 A.M.** It covers the material From Lesson 19 to Lesson L36 (Sections 5.1 to 7.5). You will not be allowed to use notes, books, calculators, etc. All you need are pencils (pens) and erasers.

The exam will have five problems. Each problem is worth 10 points. Each problem may have several parts. You may be asked to state a definition, state a theorem, judge whether a statement is true or false, or prove a statement. If you are asked for a proof, you have to give a logically correct proof written in English sentences. Scratch work is not considered a proof.

Below is a list of topics from L19 to L36 which you must know for this exam. Exam problems will be similar to quiz problems, homework problems and anything we did in class. Carefully go through your notes and homework.

A practice exam has been posted in Moodle. Treat that as a real exam. Find a nice and quiet place and then try it within the 50-minute time constraint. The **solution** will also be posted in Moodle so that you know what I expect from you.

On the day before the exam (Thursday, April 13th), I will answer your questions in an optional evening review session. **SMUD 206** has been reserved from **6:30 to 8:00 P.M.** for it.

- L19: Homotopy: Motivation
  - concatenation of two paths
  - this operation is not associative
  - definition of homotopy between two maps
  - definition of relative homotopy between two maps
  - examples
  - concatenation is associative once we consider the relative homotopy classes of maps.
  - if $f : S^1 \to S^1$ is not homotopic to $id : S^1 \to S^1$, then $f(x) = -x$ for some $x \in S^1$.
  - construct a homotopy between $f : S^1 \to S^1$ defined by $f(x) = -x$ and $id : S^1 \to S^1$

- L20: The Fundamental Group
  - definition of the fundamental group (check well-definedness of the binary operation, associativity, identity and inverse)
  - a path γ in Y connecting y_0 and y_1 induces a group isomorphism $\gamma_* : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ and its proof
  - $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ and its proof
  - $(g \circ f)_* = g_* \circ f_*$ and $id_* = id$ and their proofs
  - so a homeomorphism between two spaces induces isomorphism between the two fundamental groups

- L21: Computations: Path/Homotopy-Lifting Lemmas
  - definition of simply-connected space
  - the two proofs that the fundamental group of a topological group is simply connected
  - so $\pi_1(S^1)$ must be abelian
  - the path-lifting lemma
  - the homotopy-lifting lemma
  - outline of the proof that $\pi_1(S^1) \cong \mathbb{Z}$
  -

- L22: Computations: Unions and Products
  - computation of $\pi_1(S^2)$, $n \geq 2$ by writing S^2 as the union of two simply connected open sets whose intersection is nonempty and path-connected

  - proof that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$
  - fundamental group of the torus

- L23: A for Amherst, Deformation Retraction, Homotopy Equivalence and Contractiblity
  - definition of retraction
  - definition of deformation retraction
  - examples
  - homeomorphism, deformation retraction and homotopy equivalence (the three notations are more and more general)
  - homotopy equivalence is an equivalence relation
  - contractibility
  - examples
  - be aware that contractibility implies simple-connectedness but not vice versa
  - difference between contractibility and deformation retraction to a point

- L24: The Effect of Homotopy on f_* and thus on Homotopy Equivalent Spaces
  - recall that $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
  - given a particular f, describe what f_* does to a generator of $\pi(X, x)$
  - recall that a path γ in Y connecting y_0 and y_1 induces a group isomorphism $\gamma_* : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$
  - if $f \sim g$, then $g_* = \gamma_* \circ f_*$ where γ_* is an isomorphism of the above type
  - its proof
  - using $(g \circ f)_* = g_* \circ f_*$ and $id_* = id$, it follows that homotopy equivalent spaces have isomorphic fundamental groups
  - fundamental groups of the Möbius strip and the cylinder

- L25: The Brouwer Fixed-Point Theorem (inspired by coffee)
  - Statement of the Brouwer Fixed-Point Theorem for any n
  - the $n = 1$ case can be proved by the intermediate value theorem
  - proof of the Brouwer fixed point theorem when $n = 2$ (using the fact that retraction induces surjective homomorphism on fundamental groups)
  - Brouwer fixed point theorem doesn't hold for open disks
  - definition of fixed-point property
  - fixed-point property is a topological invariant
  - fixed-point property is not a homotopy invariant
  - fixed-point property is preserved by retraction

- L26: Another Application of "Retraction Induces Epimorphism": Surfaces, their Interiors and their Boundaries
  - recall the definition of retraction (it's different from deformation retraction)
  - retraction reduces surjective homomorphism on fundamental groups
  - definition of surface
  - definition of the interior and the boundary of a surface
  - the proof that the intersection of interior and boundary is empty
  - the proof that the Möbius trip and the cylinder are not homeomorphic
  - $\mathbb{R}^2 \ncong \mathbb{R}^3$

- L27: Simplex, Complex, Polyhedron and Triangulation
  - why do we study simplicial complexes (and simplicial maps)?
  - definition of simplex
  - definition of (simplicial) complex
  - examples
  - definition of polyhedron on a simplicial complex
  - definition of triangulation

  - definition of isomorphic simplicial complexes

- L28: Origami, Cones and Barycentric Subdivision
  - various ways of triangulating the MÖbius strip
  - triangulation of the Klein bottle
  - the cone construction
  - triangulation of the real projective plane
  - barycentric subdivision
  - iterated barycentric subdivisiion

- L29: The Key Idea: Simplicial Approximation
  - the importance of the simplicial approximation theorem
  - definition of a simplicial map between simplicial complexes
  - definition of simplicial approximation s to a continuous map f between the polyhedra of two complexes
  - s is homotopic to f (and the homotopy fixes vertices etc. by the definition of a simplicial approximation)
  - the Simplicial Approximation Theorem
  - sketch of its proof
  - the Simplicial Approximation Theorem gives an alternative proof of $\pi_1(S^2) \cong \{0\}$ if $n \geq 2$

- L30: Computing π_1, I: the Edge Group and its Convenient Presentation
  - definition of the edge group $E(K, v)$ for a simplicial complex based at vertex v
  - its relationship to $\pi_1(|K|, v)$
  - definition of the convenient presentation $G(K, L)$ where L is a maximal tree in K
  - how to compute $G(K, L)$
  - its relationship to $E(K, v)$ and thus to $\pi_1(|K|, v)$
  - so $\pi_1(|K|, v)$ only depends on the 0-, 1- and 2- simplices of $|K|$
  - thus the fundamental group of S^n is isomorphic to the fundamental group of the associated solid ball D^n where $n \geq 2$, which is the trivial group.

- L31: Computing π_1, II: The Seifert-van Kampen Theorem
  - statement of the Seifert-van Kampen Theorem
  - applications of it various examples: Klein bottle, torus, projective plane, double-holed torus etc.

- L32: The Classification of Closed Surfaces: Statement of Result
  - definition of closed surface
  - what does classification of surface mean?
  - attaching handles; attaching Möbius strips
  - the classification theorem
  - alternative statement of the classification theorem
  - $K^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2$
  - $M^2 \# T^2 \cong M^2 \# K^2$, where M is the Möbius strip
  - so $\mathbb{R}P^2 \# T^2 \cong \mathbb{R}P^2 \# K^2$

- L33: Preparation for the Proof, I: Triangulation and Orientation
  - Rad'o's result that any closed surface can be triangulated
  - the sketch of the above proof
  - definition of orientable surface
  - orientation of a 2-simplex and the induced orientation on its edges
  - compatible orientation
  - definition of orientable combinatorial surface
  - the former orientability implies the second orientability
  - definition of thickening

  - thickening of a tree gives a disc
  - thickening of a simple closed curve gives either a cylinder or a Möbius strip

- L34: Preparation for the Proof, II: Euler Characteristics
  - definition of Euler characteristic of a simplicial complex
  - examples
  - Euler characteristic of a (connected) trees
  - What can you say about the Euler characteristic of a (connected) graph?
  - maximal tree L on a combinatorial surface and its dual graph Γ
  - the relationship between the thickenings of both L and Γ
  - proof that the Euler characteristic of a closed surface is less than or equal to 2
  - Any simple closed polygonal curve separates the surface$\Leftrightarrow \chi(K) = 2 \Leftrightarrow |K| = S^2$.
  - independence of the Euler characteristic with respect to barycentric subdivision
  - $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ (used many times in the next section)

- L35: Proof, I: Surgery tells that the list is exhaustive.
  - the existence of a simply closed polygonal curve on a combinatorial surface ($\ncong S^2$) which does not separate the surface into two path-components
  - the two possible types of surgery
  - Euler characteristics of complexes triangulating circles, disks and unions of disks
  - the effect of surgery on the Euler characteristic of a surface
  - reverse the surgery procedure to recover the original surface (if the surface is orientable, the original surface is obtained by gluing to a sphere (with disks removed) a finite number of handles (cylinders gluing in the right way); if the surface is nonorientable, the original surface is obtained by gluing to a sphere (with disks removed) a finite number of Möbius strips, a finite number of handles and a finite number of cylinders in the other way.)
  - identify an arbitrary given closed surface with one on the list

- L36: Proof, II: π_1 tells us that the items on the list are different.
  - polygonal models of closed surfaces
  - uniqueness of the direct sum operation for surfaces
  - fundamental groups of all closed surfaces
  - abelianization of fundamental groups
  - conclusion of the classification theorem
  - fundamental groups of compact surfaces (possibly with boundary)

Math 455 Topology, Spring 2017
Practice Exam 2
April 14

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) Any path in S^1 is homotopic to the constant path at $1 \in S^1$. (Notice that we didn't say the end points of the path are fixed.)

   (b) (2 points) Let $|K| \cong S^2$. Then $|CK| \cong D^3$ where D^3 is the unit closed disk in $\mathbb{R}^3$.

   (c) (2 points) The Möbius strip and the the cylinder are homotopy equivalent.

   (d) (2 points) The Euler characteristic of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ is -1.

   (e) (2 points) The two groups $\langle a, b \big| a^2b^2 = 1\rangle$ and $\langle a, b \big| abab^{-1} = 1\rangle$ are isomorphic.

2. (10 points)

(a) (7 points) Prove that there is a homotopy from the map $f : S^1 \to S^1$ defined by $f(x) = -x$ to the identity map $id : S^1 \to S^1$.

(b) (3 points) Compute the fundamental group of $S^1 \times S^2$.

3. (10 points)

(a) (3 points) Prove that if A is a retraction of X, then if X has the fixed-point property, then so does A.

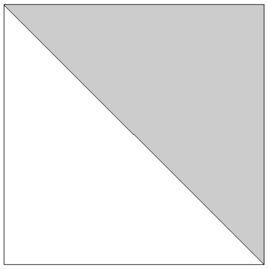
(b) (2 points) Prove that if $r : X \to A$ is a retraction, then the group homomorphism $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

(c) (5 points) Let D^2 be the closed unit disk on $\mathbb{R}^2$. Prove that any map $f : D^2 \to D^2$ has a fixed point.

4. (10 points)

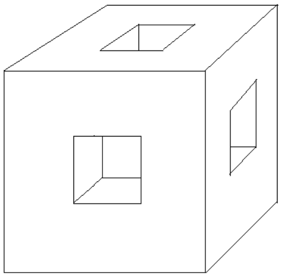
(a) (5 points) Let $K = \partial\Delta^3$. This means K consists of those simplicies of Δ^3 which are of dimension < 3. Compute $\chi(K)$.

(b) (5 points) Use $G(K, L)$ to compute the fundamental group of the polyhedron $|K|$ shown below.



5. (10 points) Two closed surfaces S_1 and S_2 are shown below.

- (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem. (Both are the surfaces you saw in the homework.)
- (3 points) Sketch their polygonal models.
- (4 points) Compute their fundamental groups.



S_1

S_2

Math 455 Topology, Spring 2017
Practice Exam 2
April 14

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) Any path in S^1 is homotopic to the constant path at $1 \in S^1$. (Notice that we didn't say the end points of the path are fixed.)

   T

   The story is different if we talk about relative homotopy

   (b) (2 points) Let $|K| \cong S^2$. Then $|CK| \cong D^3$ where D^3 is the unit closed disk in $\mathbb{R}^3$.

   T

   (c) (2 points) The Möbius strip and the the cylinder are homotopy equivalent.

   T

   (d) (2 points) The Euler characteristic of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ is -1.

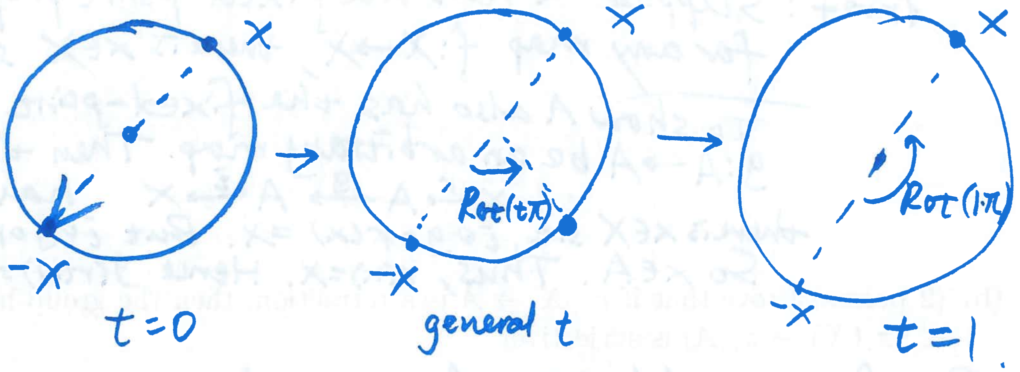
   T

   (e) (2 points) The two groups $\langle a, b \big| a^2b^2 = 1\rangle$ and $\langle a, b \big| abab^{-1} = 1\rangle$ are isomorphic.

   T

2. (10 points) The problem on the exam is not this problem. It's the one you solved in L19.

(a) (7 points) Prove that there is a homotopy from the map $f : S^1 \to S^1$ defined by $f(x) = -x$ to the identity map $id : S^1 \to S^1$.



We construct a homotopy $H: S^1 \times I \longrightarrow S^1$ ~~from~~ from $f: S^1 \to S^1$ to $id: S^1 \to S^1$ as follows:

Let $Rot(t\pi)$ be the rotation operation through angle $t\pi$ C.C.W.

Then $H(x,t) = Rot(t\pi)(-x)$

So $H(x,0) = Rot(0)(-x) = -x = f(x)$,

and $H(x,1) = Rot(\pi)(-x) = -(-x) = x = id(x)$

If you want a precise formula, let $x = (x_1, x_2)^T$ ← transpose

Then $Rot(t\pi) = \begin{bmatrix} \cos(t\pi) & -\sin(t\pi) \\ \sin(+\pi) & \cos(t\pi) \end{bmatrix}$ So $H(x,t) = \begin{bmatrix} \cos(t\pi) & -\sin(t\pi) \\ \sin(t\pi) & \cos(t\pi) \end{bmatrix} \begin{bmatrix} -x \\ -x_2 \end{bmatrix}$

(b) (3 points) Compute the fundamental group of $S^1 \times S^2$.

$$\pi_1(S^1 \times S^2) \cong \pi_1(S^1) \times \pi_2(S^2)$$

$$\cong \mathbb{Z} \times \{0\}$$

$$\cong \mathbb{Z}.$$

Recall that A is a retraction of X means: ① $A \subseteq X$.

② there is a map $r: X \to A$ such that if $i: A \hookrightarrow X$ is the inclusion, then $r \circ i = id: A \longrightarrow A$.

This simply means if $a \in X$ is actually in A. Then $r(a) = a$.

3. (10 points)

(a) (3 points) Prove that if A is a retraction of X, then if X has the fixed-point property, then so does A.

Proof: Suppose X has the fixed-point property. This mean for any map $f: X \to X$, there is $x \in X$ s.t. $f(x) = x$.

To show A also has the fixed-point property, let $g: A \to A$ be an arbitrary map. Then the map $X \xrightarrow{r} A \xrightarrow{g} A \xrightarrow{i} X$ has a fixed-point: there is $x \in X$ s.t. $i \circ g \circ r(x) = x$. But $i \circ g \circ r(x) = g \circ r(x) \in A$. So $x \in A$. Thus, $r(x) = x$. Hence $g(r(x)) = x$ implies $g(x) = x$ where $x \in A$. □

(b) (2 points) Prove that if $r : X \to A$ is a retraction, then the group homomorphism $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

Proof: By definition of retraction:

$$r \circ i = id$$

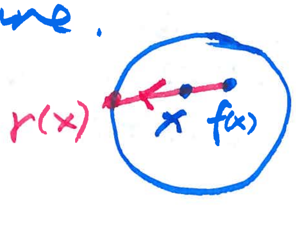
Then $(r \circ i)_* = id_*$

So $r_* \circ i_* = id_{\pi_1(A)}$.

This implies that $r_* : \pi_1(X) \to \pi_1(A)$ is surjective. □

(c) (5 points) Let D^2 be the closed unit disk on $\mathbb{R}^2$. Prove that any map $f : D^2 \to D^2$ has a fixed point.

Proof: Suppose on the contrary that for any $x \in D^2$, $f(x) \neq x$. Then define $r: D^2 \longrightarrow S^1$ as shown in the picture.



It's essential that $x \neq f(x)$ so that we can draw a line

And it's important to draw it from $f(x)$ to x.

It follows that $r(x) = x$ if $x \in S^1$. So r is a retraction. (We take it for granted that r is continuous)

By (b), $r_* : \pi_1(D^2) \longrightarrow \pi_1(S^1)$ is a surjective, which is impossible 'cause there is no surjection from $\{0\}$ to $\mathbb{Z}$. □

4. (10 points)

(a) (5 points) Let $K = \partial\Delta^3$. This means K consists of those simplicies of Δ^3 which are of dimension < 3. Compute $\chi(K)$.

Let the 0-simplices of K be v_0, v_1, v_2, v_3

Do not draw a picture. It doesn't help.

$$\chi(K) = \binom{4}{1} - \binom{4}{2} + \binom{4}{3}$$

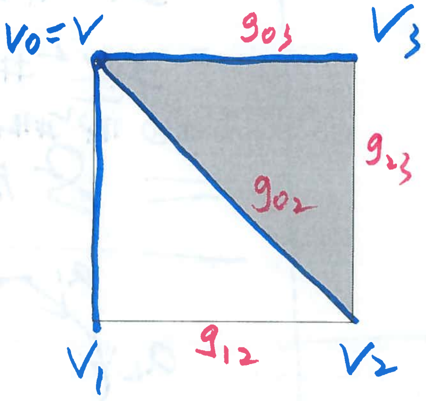
$$= 4 - 6 + 4$$

$$= 2$$

During the actual exam, you will see a higher dimensional example which can't be drawn.

The number of i-simplices is "4 choose $i+1$" $i = 0, 1, 2$

(b) (5 points) Use $G(K, L)$ to compute the fundamental group of the polyhedron $|K|$ shown below.



L is in blue, which is a maximal tree.

$$\pi_1(|K|, v) \cong E(K, v)$$

$$\cong G(K, L)$$

$$\cong \langle g_{12}, g_{23} \mid \underbrace{g_{02}}_{1} g_{23} = \underbrace{g_{03}}_{1} \rangle$$

$$\cong \langle g_{12}, g_{23} \mid g_{23} = 1 \rangle$$

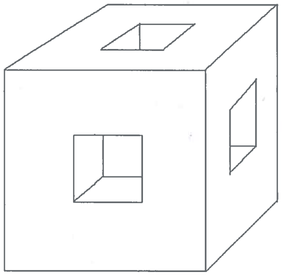
$$\cong \langle g_{12} \rangle$$

$$\cong \mathbb{Z}$$

← This makes sense since $|K| \simeq S^1$ (contract the triangle onto segment)

5. (10 points) Two closed surfaces S_1 and S_2 are shown below.

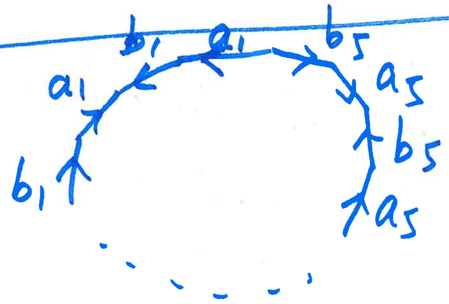
- (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem. (Both are the surfaces you saw in the homework.)
- (3 points) Sketch their polygonal models.
- (4 points) Compute their fundamental groups.



S_1

S_2

$5T^2 := T^2 \# T^2 \# T^2 \# T^2 \# T^2$



$\pi_1(5T^2)$

$\cong \langle a_1, b_1, a_2, b_2, \cdots, a_5, b_5 \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_5 b_5 a_5^{-1} b_5^{-1} = 1 \rangle$

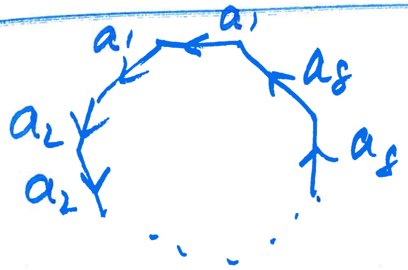
$3T^2 \# K^2$

$= 3T^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$= 3\underline{K}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$\mathbb{RP}^2 \# \mathbb{RP}^2$

$= 8\mathbb{RP}^2$



$\pi_1(\cdots)$

$\cong \langle a_1, a_2, \cdots, a_8 \mid a_1^2 a_2^2 \cdots a_8^2 = 1 \rangle$

L37

4/6 Homology: Intuitive Ideas & Introductory Examples

Motivation: homology enables us to characterize may more spaces
- ex: using π_1, we don't know if $\mathbb{R}^3 \cong \mathbb{R}^4$

↳ using homology, we find that $\mathbb{R}^m \ncong \mathbb{R}^n$ when $m \neq n$
- note: we will not go into the details in this course

Idea: Given a (triangulable) space X, we construct a sequence of objects, $H_0(X), H_1(X), \dots,$ s.t. if $X \cong Y$, then $H_i(X)$ "=" $H_i(Y)$
- $H_i(X)$ means the # of $(i+1)$D cavities bounded by iD orientable (singular) surface

Ex: $S^2 =$  : Note there is a 3D cavity bounded by the 2D closed surface
- so $H_2(S^2) \longleftrightarrow$ "1"

Ex: $S^1 =$  : Note there is a 2D cavity bounded by the 1D closed curve
- so $H_1(S^1) \longleftrightarrow$ "1"

- def: S^1 & S^2 are cycles, since they have no boundaries

Ex: a circle (1-cycle) drawn on S^2 does not bound a 2D cavity.
- so $H_1(S^2) \longleftrightarrow$ "0"

Ex: $T^2 =$  : 2 types of 1-cycles $\longrightarrow H_1(T^2) \longleftrightarrow$ "2"

Single bounded 3D cavity $\longrightarrow H_2(T^2) \longleftrightarrow$ "1"

- note:  : We think of O and O as representing the same thing

↳ since the two circles form the boundary of a subset of T^2

↳  $\rightarrow \partial M = O \cup O$: we say these 1-cycles are homologous

Ex: $\mathbb{R}P^2$ [note: this doesn't sit in $\mathbb{R}^3$, so 'cavities' aren't the best way to think]

- $H_2(\mathbb{R}P^2) \leftrightarrow$ "0"
- $H_1(\mathbb{R}P^2) \leftrightarrow$ "sits between '0' & '1'"

attributed to Henri Poincaré →

Def: These numbers are called Betti numbers

- but numbers don't work so well for $\mathbb{R}P^2$...

Emmy Noether →

↳ these $H_i(X)$ are (abelian, finitely generated) groups

Ex: (homology groups of S^2): $H_0(S^2) \cong \mathbb{Z}$, $H_1(S^2) \cong 0$, $H_2(S^2) \cong \mathbb{Z}$, $H_i(S^2) \cong 0$ when $i \geq 3$

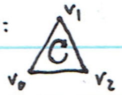
Ex: (S^1): $0 \to \mathbb{Z}$, $1 \to \mathbb{Z}$, $i \to 0$ when $i \geq 2$

Ex: (T^2): $0 \to \mathbb{Z}$, $1 \to \mathbb{Z} \oplus \mathbb{Z}$, $2 \to \mathbb{Z}$, $i \to 0$ when $i \geq 3$

Ex: $(\mathbb{R}P^2)$: $0 \to \mathbb{Z}$, $1 \to \mathbb{Z}/2\mathbb{Z}$, $i \to 0$ when $i \geq 2$

Def: X space, K complex s.t. $|K| \cong X$. We will define $H_i(K)$, $i \geq 0$

- this definition ~~used~~ uses $C_n(K)$, with elements called n-chains

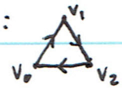
Ex: 

: $(v_0, v_1, v_2) = (v_1, v_2, v_0) = (v_2, v_0, v_1)$ is called an oriented 2-simplex

↳ note: to get from (v_0, v_1, v_2) to (v_1, v_2, v_0), we use two transpositions

- so $(v_0, v_1, v_2) = (-1)^2 (v_1, v_2, v_0)$

Def: $C_n(K) :=$ the free abelian group generated by all the n-simplices, each with an orientation chosen by you.

Ex: 

: $C_1(K) = \mathbb{Z}_{(v_0, v_1)} \oplus \mathbb{Z}_{(v_1, v_2)} \oplus \mathbb{Z}_{(v_2, v_0)} \cong \mathbb{Z}^3$

- note this is S^1

Problems for Lesson 37: Homology: Intuitive Ideas and Introductory Examples

April 6, 2017

Problem (1) will be graded.

(1) We mentioned four mathematicians' names in class today. They are Enrico Betti, Emmy Noether, Henri Poincaré and René Thom. Write a short passage about an aspect of the life or the work of one of them which interests you.

(2) Do you remember Pavel Alexandrov, the mathematician who invented one-point compactification? (You can find a picture of him in L9.) According to him (Wikipedia), Emmy Noether attended lectures given by Heinz Hopf and by him in the summers of 1926 and 1927, where she continually made observations which were often deep and subtle and when she first became acquainted with a systematic construction of combinatorial topology (an older name for algebraic topology), she immediately observed that it would be worthwhile to study directly the groups of algebraic complexes and cycles of a given polyhedron and the subgroup of the cycle group consisting of cycles homologous to zero; instead of the usual definition of Betti numbers, she suggested immediately defining the Betti group as the complementary (quotient) group of the group of all cycles by the subgroup of cycles homologous to zero. This observation now seems self-evident. But in those years (1925 – 28) this was a completely new point of view.

Emmy Noether was described by Albert Einstein et. al. as the most important woman in the history of mathematics. In her late days, She was a professor at Bryn Mawr College. After she passed away, her remains were placed near the M. Carey Thomas Library at Bryn Mawr.

L38

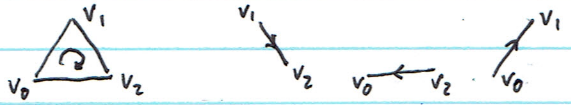
4/7 L38 Homology: Definition & First Computations

Def: The boundary homomorphism $\partial = \partial_n : C_n(K) - C_{n-1}(K)$, $(v_0, \dots, v_n)$ is a generator

$\partial(\underbrace{v_0, v_1, \dots, v_n}_{\text{oriented } n\text{-simplex}}) := \sum_{i=0}^{n} (-1)^i (v_0, \dots, \underbrace{\hat{v}_i}_{}, \dots, v_n) = (v_1, \dots, v_n) - (v_0, v_2, \dots, v_n) + \dots$

notation: v_i not there

- the general formula is obtained by linear extension:
  - say $k_1\sigma_1 + \dots + k_m\sigma_m$ is a general element in $C_n(K)$ (σ: oriented n-simplex)
  - ↳ then $\partial_n(k_1\sigma_1 + \dots + k_m\sigma_m) := k_1\partial_n(\sigma_1) + \dots + k_m\partial_n(\sigma_m)$
- note: this does correspond to "boundary"
  - ex: $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$



Def: The chain complex is the chain groups 'linked' by boundary homomorphisms.

- $\dots \xrightarrow{\partial} C_{n+1}(K) \xrightarrow{\partial} C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} C_{-1}(K) := \{0\}$

Thm: $\partial^2 := \partial_n \partial_{n+1} = 0$

Pf: Let $(v_0, v_1, \dots, v_{n+1})$ be an oriented n-simplex

Then $\partial^2(v_0, \dots, v_{n+1}) = \partial_n \partial_{n+1}(v_0, \dots, v_{n+1})$

$= \partial_n \left[\sum_{i=0}^{n+1} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_{n+1}) \right]$

$= \sum_{i=0}^{n+1} (-1)^i \partial_n (v_0, \dots, \hat{v}_i, \dots, v_{n+1})$ [∂ linear]

$= \sum_{i=0}^{n+1} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j (v_0, \dots, \hat{v}_j, \dots, v_{n+1}) + \sum_{j=i+1}^{n+1} (-1)^{j-1} (\dots) \right]$ $(v_0, \dots, \hat{v}_j, \dots, v_{n+1})$

Consider one term in sum: $(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1})$

It shows up twice, with signs $(-1)^{i+j}$ and $(-1)^{i+j-1}$

So $\partial^2(v_0, \dots, v_{n+1}) = 0$ □

an element is an n-cycle

Def: The n-cycle group $Z_n(K) := \mathrm{Ker}\,\partial_n = \{ c \in C_n(K) \mid \partial_n(c) = 0 \}$

Def: The n-boundary group $B_n(K) := \mathrm{Im}\,\partial_{n+1} = \{ \partial_{n+1}(c) \in C_n(K) \mid c \in C_{n+1}(K) \}$

an element is an n-boundary

Note: $Z_n(K), B_n(K)$ are subgroups of $C_n(K)$

Prop: $B_n(K)$ is a subgroup of $Z_n(K)$ (i.e. $B_n(K) < Z_n(K) < C_n(K)$)

Pf: $\partial_n B_n(K) = \partial_n \partial_{n+1} C_{n+1}(K) = 0$ [previous thm] □

- i.e. any n-boundary is an n-cycle

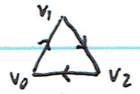
next is $(\mathbb{Z}/5\mathbb{Z})^{n_3} \dots$

Def: The n^{th} homology group $H_n(K) := Z_n(K) / B_n(K)$

- note: $H_n(K) = \underbrace{\mathbb{Z}^{\boxed{\beta_i}}}_{\text{free part}} \oplus \underbrace{(\mathbb{Z}/2\mathbb{Z})^{n_1} \oplus (\mathbb{Z}/3\mathbb{Z})^{n_2} + \dots}_{\text{torsion part}}$ (finitely many) (primes)

Betti number

Ex: K =



: $C_1(K) \cong \mathbb{Z}^3$ (generated by $(v_0,v_1),(v_1,v_2),(v_2,v_0)$)

$C_0(K) \cong \mathbb{Z}^3$ (generated by v_0, v_1, v_2)

$|K| \simeq S^1$ $\quad C_i(K) = 0$ for $i \geq 2$

$$\longrightarrow \dots \to \overset{=0}{C_3(K)} \to \overset{=0}{C_2(K)} \to C_1(K) \to C_0(K) \to 0$$

- clearly, $H_i(K) \cong 0$ for $i \geq 2$
- $H_1(K) = Z_1(K) / B_1(K) = \mathrm{Ker}\,\partial_1 / \mathrm{Im}\,\partial_2 = \mathrm{Ker}\,\partial_1 \cong \mathbb{Z}$

Pf: let $k_1(v_0,v_1) + k_2(v_1,v_2) + k_3(v_2,v_0) \in C_1(K)$ s.t. $\partial(\dots) = 0$

- so $k_1 \partial(v_0,v_1) + k_2 \partial(v_1,v_2) + k_3 \partial(v_2,v_0) = 0$
- so $k_1(v_1 - v_0) + k_2(v_2 - v_1) + k_3(v_0 - v_2) = 0$
- so $(-k_1 + k_3)v_0 + (k_1 - k_2)v_1 + (k_2 - k_3)v_2 = 0$

We know v_0, v_1, v_2 are generators, so each $(-k_1 + k_3), \dots = 0$.

So $k_1 = k_2 = k_3$, i.e. 1 generator.

- $H_0(K) = \mathrm{Ker}\,\partial_0 / \mathrm{Im}\,\partial_1 = C_1(K) / \mathrm{Im}\,\partial_1 \cong \mathbb{Z}$

Pf: $\partial(v_0, v_1) = v_1 - v_0$: v_0, v_1 represent the same class

similar for $\partial(v_2, v_0)$ → only one generator

Problems for Lesson 38: Homology: Definition and First Computations

April 7, 2017

Problem (1) will be graded.

1. (a) Let K be the complex triangulating S^1 which has four vertices. Compute $H_i(K)$, $i = 0, 1, 2, \cdots$

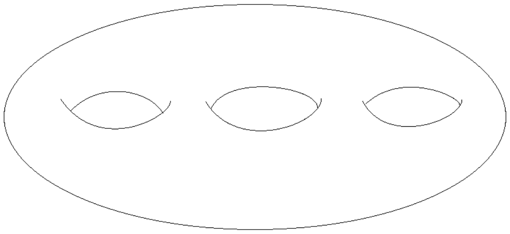
   (b) Let K be the complex triangulating S^2 which consists of all proper faces (faces of dimension < 3) of Δ^3. Compute $H_i(K)$, $i = 0, 1, 2, 3 \cdots$

2. Check again that $\partial^2 = 0$.

MATH 455 Quiz #10 Name:____________________

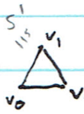
A closed surface is shown below.

1. (3 points) Identify the surface with a standard surface on the list of the classification theorem.
2. (3 points) Sketch its polygonal model.
3. (4 points) Compute its fundamental group. (just write the answer)



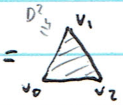
4/10 Homology of Cones (and thus of spheres)

or $[v_1], [v_2]$

Recall: $K =$ 

: $H_0(K) \cong \mathbb{Z}$ (generated by $[v_0]$)
$H_1(K) \cong \mathbb{Z}$ (generated by $(v_0,v_1)+(v_1,v_2)+(v_2,v_0)$)

∘ and $H_i(K) \cong 0$ when $i \geq 2$, since $C_i(K) = 0$ when $i \geq 2$

Ex: $L =$ 

: $H_0(L) \cong \mathbb{Z}$ [L38, hw]
$H_i(L) \cong 0$ when $i \geq 1$

∘ note: chain complex: $\ldots 0 \to C_2(L) \to \underbrace{C_1(L) \to C_0(L)}_{\text{same as prev. example}} \to 0$

Thm: K complex. Then:

: 3 path-components.

① $H_0(K) \cong \mathbb{Z}^c$, where c is the # of path-components of K.

② if $|K|$ is path-connected, then $H_0(K) \cong \mathbb{Z}$ $(c=1)$

Pf: (②) (① is similar)

$H_0(K) = Z_0(K)/B_0(K)$

$= C_0(K)/\partial_1 C_1(K)$ [chain complex: $C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$]

Note that $C_0(K)$ is generated by the 0-simplices of K.

Pick one such 0-simplex v_0 in K.

Then ∀ other 0-simplices v_*, there is edge path $v_0 v_{i_1} \cdots v_{i_k} v_*$ in K [path-connected]

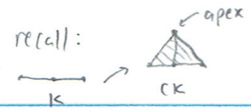
So $\partial_1(\underbrace{(v_0,v_{i_1}) + \ldots + (v_{i_k},v_*)}_{\in C_1(K)}) = v_{i_1} - v_0 + v_{i_2} - v_{i_1} + \ldots + v_* - v_{i_k} = v_* - v_0$

So $v_* - v_0 \in \partial_1 C_1(K) =: B_0(K)$, i.e. $[v_*] = [v_0]$

Furthermore, $v_0 \notin B_0(K)$ (i.e. nonempty H_0)

So $H_0(K) \cong \mathbb{Z}$ (generated by $[v_0]$) □

recall:



Thm: K complex. CK is cone over K. Then $H_0(CK) \cong \mathbb{Z}$, $H_i(CK) \cong 0$ for $i \geq 1$

Pf: Note that CK is always path-connected (even if K is not).

So $H_0(CK) \cong \mathbb{Z}$ [prev. thm]

Now define group homomorphisms d as follows:

$$\begin{tikzcd} \cdots \arrow[r, shift left, "\partial"] & C_{n+1}(CK) \arrow[l, shift left, "d"] \arrow[r, shift left, "\partial"] & C_n(CK) \arrow[l, shift left, "d"] \arrow[r, shift left, "\partial"] & \cdots \arrow[l, shift left, "d"] \arrow[r, shift left, "\partial"] & C_1(CK) \arrow[l, shift left, "d"] \arrow[r, shift left, "\partial"] & C_0(CK) \arrow[l, shift left, "d"] \arrow[r, "\partial"] & 0 \end{tikzcd}$$

no d.

◦ ex: $d: C_n(CK) \to C_{n+1}(CK)$ by $d(\underbrace{v_0, \dots, v_n}_{\sigma}) = \begin{cases} (v, v_0, \dots, v_n) & \text{if } \sigma \in K \\ 0 & \text{if } \sigma \in CK \setminus K \end{cases}$

- general d is obtained by linear extension
- note: $d(-\sigma) = -d(\sigma)$, analogous to $\partial(-\sigma) = -\partial(\sigma)$

so $\partial d + d\partial = id$

Then $\partial d(\sigma) = \sigma - d\partial(\sigma)$ [hw, 1b]. (half of pf in textbook)

2 case: $\sigma \in K$
$\sigma \in CK \setminus K$

Then, if $z \in Z_n(CK)$, $n \geq 1$, $\partial d(z) = z - d\underbrace{\underbrace{\partial(z)}_{0}}_{0}$

So $z = \partial(dz) \in B_n(CK)$

So $H_n(CK) := Z_n(CK) / B_n(CK) \cong 0$ □

note: this doesn't work for H_0, since that d is not defined

(abuse of notation)

Cor: Let Δ^{n+1} be the complex whose simplices are all the faces of standard simplex Δ^{n+1}.

Then $\Delta^{n+1} = C\Delta^n$, so $H_0(\Delta^{n+1}) \cong \mathbb{Z}$, $H_i(\Delta^{n+1}) \cong 0$ when $i \geq 1$ □

Thm: Let $n \geq 1$. Let $\partial\Delta^{n+1}$ be the complex whose simplices are the faces of Δ^{n+1} except that single $(n+1)$-complex [so $|\partial\Delta^{n+1}| \cong S^n$]. Then $H_0(\partial\Delta^{n+1}) \cong \mathbb{Z}$, $H_n(\partial\Delta^{n+1}) \cong \mathbb{Z}$, $H_i(\partial\Delta^{n+1}) \cong 0$ ow.

Pf: Consider

$$\begin{array}{r} \cdots \to 0 \to C_{n+1}(\Delta^{n+1}) \to \\ \cdots \to 0 \to C_{n+1}(\partial\Delta^{n+1}) \to \end{array} \boxed{\begin{array}{l} C_n(\Delta^{n+1}) \to \cdots \to C_0(\Delta^{n+1}) \to 0 \\ C_n(\partial\Delta^{n+1}) \to \cdots \to C_0(\partial\Delta^{n+1}) \to 0 \end{array}}$$

identical!
◦ share simplices (only n+1-simplex is diff.)

So $H_i(\partial\Delta^{n+1}) \cong H_i(\Delta^{n+1})$ $0 \leq i \leq n-1$

0

Now $H_n(\partial\Delta^{n+1}) := Z_n(\partial\Delta^{n+1}) / B_n(\partial\Delta^{n+1}) = Z_n(\partial\Delta^{n+1}) = Z_n(\Delta^{n+1})$

$\cong B_n(\Delta^{n+1}) := \partial \underbrace{C_{n+1}(\Delta^{n+1})}_{\mathbb{Z}} \cong \mathbb{Z}$

$H_n(\Delta^{n+1}) \cong 0$

Problems for Lesson 39: Homology of Cones and thus of Spheres

April 10, 2017

Problem (1) will be graded.

(1) (a) From what we proved in class, because $\Delta^2 = C\Delta^1$, we then have $H_0(\Delta^2) \cong \mathbb{Z}$ and $H_i(\Delta^2) \cong 0$ for all $i \geq 1$. Prove the same result (compute $H_i(\Delta^2)$ for $i \geq 0$) using definition of homology instead.

(b) Let K be an arbitrary simplicial complex and CK the cone over it with apex v. For the group homomorphism $d : C_n(CK) \to C_{n+1}(CK)$, $n \geq 0$, defined on generators $\sigma = (v_0, v_1, \cdots, v_n)$ by

$$d(\sigma) = \begin{cases} (v, v_0, v_1, \cdots, v_n) & \text{if } \sigma \in K, \\ 0 & \text{if } \sigma \in CK \backslash K, \end{cases}$$

prove that

$$\partial d(\sigma) = \sigma - d\partial(\sigma),$$

(which we used in class to show that all homology groups of degree > 0 of a cone are trivial.)

(2) Hopefully you have checked $\partial^2 = 0$ again (on your own). Recall that this is the reason we can define homology groups. Below is a quote on the first page of the classic **Sergei I. Gelfand, Yuri I. Manin**, *Methods of Homological Algebra*, Springer, Berlin and Heidelberg, 1997, 2003. Manin is my Ph.D. advisor Ralph M. Kaufmann's Ph.D. advisor at University of Bonn.

... utinam intelligere possim rationacinationes pulcherrimas quae e propositione concisa DE QUADRATUM NIHILO EXAEQUARI fluunt.

(... if I could only understand the beautiful consequence following from the concise proposition $d^2 = 0$.)

From Henri Cartan Laudatio on receiving the degree of Doctor Honoris Causa, Oxford University, 1980.

(3) In fact, you saw $d^2 = 0$ even when you were a freshmen or sophomore (or high school student). In MATH 211, you probably learned gradient, curl and divergence, which are three differential operations. (Let d be each of them.) Show that

$$\text{curl} \circ \text{grad} = 0$$

and

$$\text{div} \circ \text{curl} = 0.$$

These are two theorems in Stewart.

4/12 Homology of Surfaces

Thm: Let K be a combinatorial closed surface. Then $H_0(K) \cong \mathbb{Z}$

Pf: K connected $\to H_0(K) \cong \mathbb{Z}$ [L39, theorem 1.2] □

Note: $H_i(K) \cong 0$ for $i \geq 3$, since $C_i(K) \cong 0 \to C_i(K) > Z_i = 0$

Note: $H_1(K)$ could be computed by brute force, but it'd take a long time

(any-dimensional) abelianization

Thm: Let K be arbitrary complex s.t. $|K|$ is connected. Then $H_1(K) \cong$ ab. of $\pi_1(|K|)$

Cor: Let K be combinatorial closed surface. Then $H_1(K) \cong 0$ if $|K| \cong S^2$, $H_1(K) \cong \mathbb{Z}^{2m}$ if $|K| \cong mT^2$, and $H_1(K) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}^{n-1}$ if $|K| \cong n\mathbb{R}P^2$

Pf: (of thm, sketch)

Let v be a vertex in K. Then $\pi_1(|K|, v) \cong E(K, v)$

Define $\Phi: E(K,v) \to H_1(K)$ by $\Phi(\{\alpha\}) = [z(\alpha)]$, where

α is an edge loop $vv_1 \cdots v_k v$, $\{\alpha\}$ is its equivalence class

$z(\alpha) := (v, v_1) + (v_1, v_2) + \ldots + (v_k, v)$. Note this is a 1-cycle.

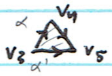
Pf: $\partial z(\alpha) = -v + v = 0$

~~$\cdots v_1 v_1 \cdots$~~

Note: representation α is chosen s.t. it has no repeated vertices.

Φ is well-defined

Ex: Let α & α'. They differ by:



So $(v_3, v_4) + (v_4, v_5) = (v_3, v_5) + \partial(v_3, v_4, v_5)$

So $[z(\alpha)] = [z(\alpha')]$ [only differ by boundary element]

(the general case is similar) □

Φ is a group homomorphism

Pf: By definition: $\Phi(\{\alpha\} \cdot \{\beta\}) = \Phi(\{\alpha\})\,\Phi(\{\beta\})$

Φ is onto

you can shuffle elems.

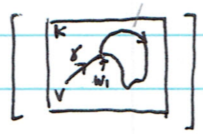
Pf: Pick any 1-cycle $z = (w_1, w_2) + \ldots + (w_{k-1}, w_k)$ in $Z_1(K)$

Let γ be an edge path in K from v to w_1.

Then $\Phi\{\gamma\beta\gamma^{-1}\} = [z(\gamma\beta\gamma^{-1})] = [z]$

[1-simplices on γ, γ^{-1} cancel out] □

call loop from z β.



$\longrightarrow$

Consider elements in $G(K,L)$.
Then $\{\alpha\beta^{-1}\}$ is product
of elements of $a,b,c,d,\ldots$
s.t. elements show up in pairs ($aaaa^{-1}a^{-1}a^{-1}$, e.g.)

Thm: $H_1(K) \cong$ abelianization of $\pi_1(|K|)$

Pf: (cont'd)

Since Φ is an onto homomorphism, Φ induces: $E(K,v)/\operatorname{Ker}\Phi \cong H_1(K)$

We will show $\operatorname{Ker}\Phi = [E(K,v),E(K,v)]$

$[E(K,v),E(K,v)]$ is the commutator subgroup of $E(K,v)$. Its elements are finite products of terms in this form: $[a,b] := aba^{-1}b^{-1}$

Pf: Since $H_1(K)$ is abelian by definition, it must be true that $[E(K,v),E(K,v)] \subseteq \operatorname{Ker}\Phi$

Pf: LTR (pf by contradiction)

$\operatorname{Ker}\Phi \subseteq [E(K,v),E(K,v)]$

Pf: LTR (in book) - sketch below.

Let $\{\alpha\} \in \operatorname{Ker}\Phi$

This means that its image under Φ, $[z(\alpha)] = 0 \in H_1(K)$

So $z(\alpha) \in B_1(K)$

Then there are $n_1,\ldots,n_\ell \in \mathbb{Z}$ and $\sigma_1,\ldots,\sigma_\ell$ oriented 2-simplices in $C_2(K)$, s.t. $z(\alpha) = \partial(n_1\sigma_1 + \ldots + n_\ell\sigma_\ell)$

Using those n_i and σ_i, we can produce edge path β s.t. $\{\alpha\} = \{\alpha\beta^{-1}\}$ and $z(\alpha\beta^{-1}) = 0$.

Now consider elements in $G(K,L)$.

The above means $\{\alpha\beta^{-1}\}$ is a product of elements of $a,b,c,\ldots$ s.t. e.g. $aaaa^{-1}a^{-1}a^{-1}$ (elements show up in pairs).

So $\{\alpha\beta^{-1}\} \in [E(K,v),E(K,v)]$ □

QED thm.

Thm: $H_2(K) \cong \mathbb{Z}$ if K is orientable, $H_2(K) \cong 0$ if K is nonorientable

Pf: Consider $\partial(n_1\sigma_1 + \ldots + n_k\sigma_k) = 0$

↳ orientable: $n_1 - n_2 = 0, \ldots, n_{k-1} - n_k = 0, n_k - n_1 = 0 \Rightarrow n_1 = \ldots = n_k$

↳ nonorientable: $n_1 - n_2 = 0, \ldots, n_{k-1} - n_k = 0, n_k + n_1 = 0 \Rightarrow n_1 = \ldots = n_k = 0$. □

Problems for Lesson 40: Homology of Surfaces

April 12, 2017

Problem (2) will be graded.

(1) Convince yourself that $H_2(K) \cong \mathbb{Z}$ if K is an orientable combinatorial closed surface while $H_2(K) \cong 0$ is K is a non-orientable combinatorial closed surface.

(2) Let $K_{m,r}$ be a complex whose polyhedron is obtained by removing the interior of $r \geq 1$ disjoint closed discs from mT^2. Let $L_{n,r}$ be a complex whose polyhedron is obtained by removing the interior of $r \geq 1$ disjoint closed discs from $n\mathbb{R}P^2$. Compute $H_i(K_{m,r})$ and $H_i(L_{n,r})$ for all $i = 0, 1, 2, \cdots$

Comment: Problem (1)(b,c) from L36 is useful.

4/13 Chain Maps between Chain Complexes

Def: K complex. The chain complex of K is $C_\cdot(K) := \dots \xrightarrow{\partial} C_n(K) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$

Def: K, L complexes. $C_\cdot(K)$, $C_\cdot(L)$ chain complexes. A chain map $\phi_\cdot : C_\cdot(K) \to C_\cdot(L)$ is a sequence of gp homomorphisms $\phi_n : C_n(K) \to C_n(L)$, $n = 0, 1, \dots$ s.t. for each $n \geq 0$, $\phi_n \circ \partial_{n+1} = \partial_{n+1} \circ \phi_{n+1}$

- visually:

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1}(K) \arrow[r, "\partial_{n+1}"] \arrow[d, "\phi_{n+1}"] & C_n(K) \arrow[r] \arrow[d, "\phi_n"] & \dots \\ \dots \arrow[r] & C_{n+1}(L) \arrow[r, "\partial_{n+1}"] & C_n(L) \arrow[r] & \dots \end{tikzcd}$$

: travelling ↴ is the same as travelling ↳ along the grid
- in image, denoted as ✱ or ↻

- note: by convention, we write ∂_n, ϕ_n as ∂, ϕ → $\phi \circ \partial = \partial \circ \phi$ [different ϕ]

Def: Given a chain map $\phi_n = \phi : C_n(K) \to C_n(L)$, then define $\phi_* : H_n(K) \to H_n(L)$ by $\phi_*([z]) = [\phi(z)]$

- Note: We know $z \in Z_n(K)$. Then $\phi(z) \in Z_n(L)$.

φwell-def.

Pf: We have $\phi \circ \partial = \partial \circ \phi$.

So $\underbrace{\phi \circ \underbrace{\partial(z)}_{0}}_{0} = \partial \circ \phi(z) \implies \partial(\phi(z)) = 0$, so $\phi(z) \in Z_n(L)$

← boundary of cycle is 0

← gp homomorphisms send $0 \to 0$

- Note: If $[z] = [w]$, then $[\phi(z)] = [\phi(w)]$

Pf: $[z] = [w]$ means $\exists u \in C_{n+1}(K)$ s.t. $z - w = \partial u \in B_n(K)$

So $\phi(z-w) = \phi \circ \partial u \implies \phi(z) - \phi(w) = \partial \circ \phi(u) = \partial \overbrace{\phi(u)}^{\in C_{n+1}(L)} \in B_n(L)$

So $[\phi(z)] = [\phi(w)]$ in $H_n(L)$ [differ by boundary element]

- Note: ϕ_* is a homomorphism b/c ϕ is one.

Thm: Given $\phi : C_n(K) \to C_n(L)$, $\psi : C_n(L) \to C_n(M)$ chain maps, then:

① $\psi \circ \phi$ is a chain map

② $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

Pf: (①) $(\psi \circ \phi) \circ \partial = \psi \circ (\phi \circ \partial) = \psi \circ (\partial \circ \phi) = (\psi \circ \partial) \circ \phi = (\partial \circ \psi) \circ \phi = \partial \circ (\psi \circ \phi)$

Since ψ, ϕ are group homomorphisms, so $\psi \circ \phi$ is too.

- visually:

$$\begin{tikzcd}
C_{n+1}(K) \arrow[r, "\partial"] \arrow[d, "\phi"'] & C_n(K) \arrow[d, "\phi"'] \\
C_{n+1}(L) \arrow[r, "\partial"] \arrow[d, "\psi"'] & C_n(L) \arrow[d, "\psi"'] \\
C_{n+1}(M) \arrow[r, "\partial"] & C_n(M)
\end{tikzcd}$$

: "stepping down" thru commutative squares

↳ ⤵ → ↴ → ↳

(②) $(\psi \circ \phi)_*([z]) := [\psi \circ \phi(z)]$

Also, $\psi_* \circ \phi_*([z]) := \psi_*(\phi_*([z])) := \psi_*([\phi(z)]) := [\psi(\phi(z))]$

So $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

Problems for Lesson 41: Chain Maps between Chain Complexes

April 13, 2017

Problem (1) will be graded.

(1) Let K and L be complexes and $s : |K| \to |L|$ a simplicial map. (Simplicial map was introduced in L29.) Recall that s is determined by what it does on vertices. The rest is linear extension on each higher dimensional simplex. Now we construct homomorphism $s_n : C_n(K) \to C_n(L)$, $n = 0, 1, 2, \cdots$ by specifying what it does on generators as follows.

Let $\sigma = (v_0, v_1, v_2 \cdots, v_n) \in C_n(K)$. Then

$$s_n(\sigma) := \begin{cases} (s(v_0), s(v_1), s(v_2), \cdots, s(v_n)) & \text{if all } s(v_0), s(v_1), s(v_2) \cdots, s(v_n) \text{ are distinct;} \\ 0 & \text{if for some } i \neq j, s(v_i) = s(v_j). \end{cases}$$

Show that these s_n form a chain map. (So from what we did in class, it follows that s_n induces a homomorphism from $H_n(K)$ to $H_n(L)$.)

Hint: The solution can be found from Page 184 to 185 in the textbook. But surely, at least try it on your own first.

Math 455 Topology, Spring 2017
Exam 2
April 14

You are not allowed to use books, notes or calculators. You must explain your answers completely and clearly to get full credit.

Name:

1. (10 points) For the following problems, just write T or F.

   (a) (2 points) Let A be a retraction of X. Then if X has the fixed-point property, then so does A.

   (b) (2 points) Let $|K| \cong S^1$. Then $|CK| \cong D^2$ where D^2 is the unit closed disk in $\mathbb{R}^2$.

   (c) (2 points) We need at least 10 triangles to find a triangulation of the Möbius strip in $\mathbb{R}^3$.

   (d) (2 points) The Euler characteristic of $T^2 \# T^2 \# T^2$ is -4.

   (e) (2 points) The two groups $\langle a, b \big| a^2 b^2 = 1 \rangle$ and $\langle a, b \big| aba^{-1}b^{-1} = 1 \rangle$ are isomorphic.

2. (10 points)

   (a) (7 points) Prove that if the map $f : S^1 \to S^1$ is not homotopic to the identity map $id : S^1 \to S^1$, then there is $x \in S^1$ such that $f(x) = -x$.

   (b) (3 points) Compute the fundamental group of $S^1 \times S^1 \times S^1$.

3. (10 points)

   (a) (2 points) Let A be a subspace of X and $r : X \to A$ a map. What does it mean to say r is a retraction from X to A?

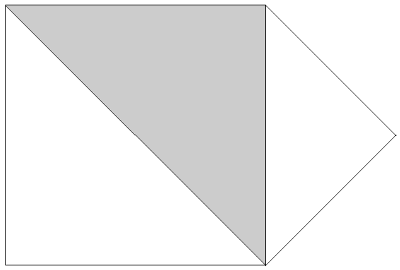
   (b) (3 points) Prove that if $r : X \to A$ is a retraction, then the group homomorphism $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

   (c) (5 points) Let D^2 be the closed unit disk on $\mathbb{R}^2$. Prove that any map $f : D^2 \to D^2$ has a fixed point.

4. (10 points)

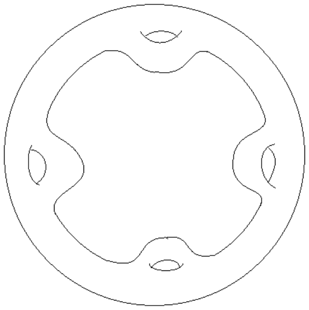
(a) (5 points) Let $K = \partial\Delta^4$. This means K consists of those simplicies of Δ^4 which are of dimension < 4. Compute $\chi(K)$.

(b) (5 points) Use $G(K, L)$ to compute the fundamental group of the polyhedron $|K|$ shown below.

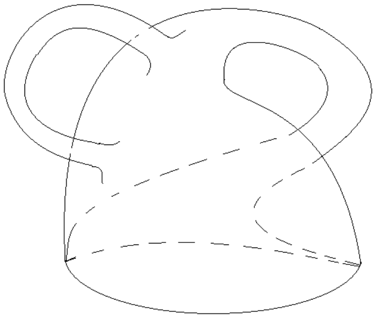


5. (10 points) Two closed surfaces S_1 and S_2 are shown below.

- (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem.
- (3 points) Sketch their polygonal models.
- (4 points) Compute their fundamental groups.



S_1



S_2

MATH 455 Quiz #11 Name:____________________

(1) (2 points) True or False? Let K be a complex and CK the cone over K. Then $H_0(CK) \cong \mathbb{Z}$ and $H_i(CK) \cong 0$ for all $i > 0$.

(2) (2 points) Ture or False? For any $n \geq 1$, $H_n(\partial\Delta^{n+1}) \cong \mathbb{Z}$.

(3) (2 points) True or False? For a complex K triangulating S^2, $B_2(K) = 0$.

(4) (2 points) True or False? If $|K|$ has three path components, then $H_0(K) \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$.

(5) (2 points) Ture or False? A chain map $\phi_\bullet : C_\bullet(K) \to C_\bullet(L)$ is defined to be any sequence of group homomorphisms $\phi_n : C_n(K) \to C_n(L)$.

4/17 Homotopy Invariance of Homology, I: Barycentric Subdivision & Stellar Subdivisions

Thm: $H_n(K) \cong H_n(K^m)$ $\forall n$

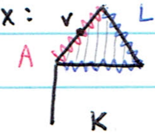
Pf: [Step 1: break barycentric subdivisions into stellar subdivisions
Step 2: show $H_n(K) \cong H_n(K')$, iterate m times]

Def: Let A be a simplex of K, v be the barycenter of A. K' is formed from K by a stellar subdivision w.r.t. A if K' is obtained as follows:

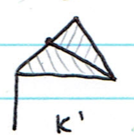
① we do nothing to those simplices in K which don't have A as a face

② If a simplex B in K has A as a face, let L be the subcomplex of the boundary of B s.t. the simplices in L do not have A as a face.
↳ Now replace B by cone CL having v as the apex

Ex:

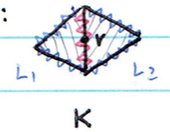


B is the 2-simplex
: L is $\partial B \setminus A$ ⟹

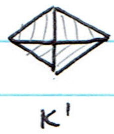


(1 stellar subdiv. wrt A)

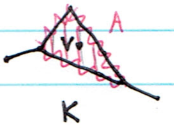
Ex:



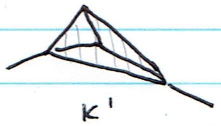
: B_1 is left 2-simplex
B_2 is right 2-simplex ⟹



Ex:



B is A
: L is boundary triangle ⟹
(△)

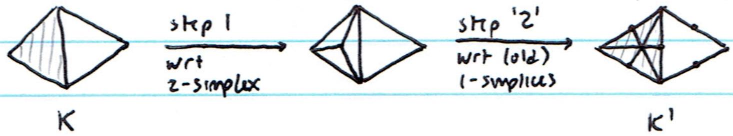


Ex:

: A = K
B = A ⟹

Fact: Given complex K, if we stellar subdivide K wrt each simplex (dim ≥ 1) in the order of decreasing dimensions, then we get K'

• ex:



$\forall n$

Thm: If K' is obtained from K by one stellar subdivision, then $H_n(K') \cong H_n(K)$

Cor: Iteratively applying previous fact & theorem $\Rightarrow H_n(K') \cong H_n(K)$
and so $H_n(K^m) \cong H_n(K)$

Pf: (of thm, sketch)

We define two chain maps:

① The subdivision chain map $\chi : C_n(K) \to C_n(K')$

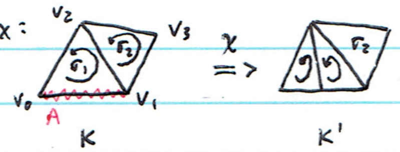
② (Related to) the standard simplicial map $\theta : C_n(K') \to C_n(K)$

Def. of ①: Let $\sigma = (v_0, \dots, v_k, v_{k+1}, \dots, v_n) \in C_n(K)$

ⓐ If $v_0, \dots, v_k$ are the vertices of A (chosen simplex), then $\chi(\sigma) := \sum_{i=0}^{k} (-1)^i (v, v_0, \dots, \hat{v}_i, \dots, v_n)$

ⓑ If A is not a face of σ, then $\chi(\sigma) := \sigma$

Ex:



Consider $\chi : C_2(K) \to C_2(K')$

: ∘ $\chi(\sigma_1) = \chi(\overbrace{v_0, v_1}^{A}, v_2) = (v, v_1, v_2) - (v, v_0, v_2)$ $+ (v, v_2, v_0)$

∘ $\chi(\sigma_2) = \sigma_2$

Fact: χ is a chain map.

Def. of ②: Let $\theta : |K'| \to |K^*|$ be the simplicial map defined by sending $v \mapsto v_0$ and fixing all other vertices. Also, let θ denote the induced chain map $\theta : C_n(K') \to C_n(K)$ $\forall n$ [see L41, hw].

Fact: $\theta \circ \chi = id_{C_n(K)}$. So $(\theta \circ \chi)_* = (id_{C_n(K)})_* \Rightarrow \theta_* \circ \chi_* = id_{H_n(K)}$

Prop: Even though $\chi \circ \theta \neq id_{C_n(K)}$, $\chi_* \circ \theta_* = id_*$ holds.

Pf: Follows from fact that $H_n(\text{cone}) \cong 0$ unless $n = 0$.

So $H_n(K) \underset{\theta_*}{\overset{\chi_*}{\rightleftarrows}} H_n(K')$ $\forall n$, i.e. $H_n(K) \cong H_n(K')$ □

Problems for Lesson 42: Homotopy Invariance of Homology, I: Barycentric Subdivision is a sequence of Stellar Subdivisions

April 17, 2017

Problem (1) will be graded.

(1) Let K be a simplicial complex and A a simplex in K. Let K' be the simplicial complex obtained from K by stellar-subdivision with respect to A. Let $\chi : C_n(K) \to C_n(K')$ be the subdivision chain map. Verify that if $\sigma = (v_0, v_1, v_2, v_3, v_4)$ is an oriented simplex in K and v_0, v_1, v_2 are the vertices of A, then $\partial \circ \chi(\sigma) = \chi \circ \partial(\sigma)$.

(2) Prove that $\chi_* \circ \theta_* = id_{H_n(K')}$ for all n.

Hint: See Page 188.

4/19 Homotopy Invariance of Homology, II: Sketch of Proof & Applications

Thm: Let K, L be complexes s.t. $|K| \overset{\text{homotopy equiv.}}{\simeq} |L|$. Then $H_n(K) \cong H_n(L)$ $\forall n$.

Cor: In particular, if X has two triangulations K & L, i.e. $|K| \cong X \cong |L|$, then $H_n(K) \cong H_n(L)$ $\forall n$.

- i.e. homology does not depend on triangulation

Def: Given triangulable space X, $H_n(X) := H_n(K)$ where K is complex s.t. $|K| \cong X$.

Pf: (of thm, ~~sketch~~ ideas) Follows from 3 facts:

(1) Fact: Any $f: |K| \to |L|$ induces a group homomorphism $f_*: H_n(K) \to H_n(L)$ $\forall n$.

(2) Fact: If $|K| \xrightarrow{f} |L| \xrightarrow{g} |M|$, then $(g \circ f)_* = g_* \circ f_*$ and $f = id_{|K|} \Rightarrow f_* = id_{H_n(K)}$ $\forall n$.

(3) Fact: If $f, g: |K| \to |L|$ are homotopic ($f \simeq g$), then $f_* = g_*$

Key concepts in proof:

- Barycentric subdivision, simplicial approximation, stellar subdiv χ, stnd. simp. map θ
- Break homotopy into a sequence of simp. approx. where adjacent simp. maps are "close"
- Chain homotopy
- Lebesgue Lemma

Pf: (of thm, sketch)

$g \circ f \simeq id_{|K|}$, $f \circ g \simeq id_{|L|}$

Let $f: |K| \to |L|$ and $g: |L| \to |K|$ be homotopy inverses.

Then $(g \circ f)_* = (id_{|K|})_*$ [F1, F3]

So $g_* \circ f_* = id_{H_n(K)}$ [F2]

Similarly, $f_* \circ g_* = id_{H_n(L)}$

So f_*, g_* are bijections, so they are isomorphisms □

Prop: For each $n>0$, $H_0(S^n) := H_0(\partial\Delta^{n+1}) \cong \mathbb{Z}$, $H_n(S^n) := H_n(\partial\Delta^{n+1}) \cong \mathbb{Z}$, and $H_i(S^n) := H_i(\partial\Delta^{n+1}) = 0$ if $i \neq 0, n$. For $n=0$, $\underbrace{H_0(S^0)} \cong \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ and $H_i(S^0) = 0$ if $i>0$.

Cor: $\mathbb{R}^m \cong \mathbb{R}^n \iff m=n$, $m,n \geq 1$.

Pf: ($\Leftarrow$) trivial.

($\Rightarrow$) Suppose $\mathbb{R}^m \xrightarrow[f]{\cong} \mathbb{R}^n$.

Then $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{f(0)\}$

We know $\mathbb{R}^m \setminus \{0\}$ deformation retracts to S^{m-1} [radial retraction]

So $\mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$

Also, $\mathbb{R}^n \setminus \{f(0)\} \overset{\text{translation}}{\cong} \mathbb{R}^n \setminus \{0\} \xrightarrow[\text{ret.}]{\text{def.}} S^{n-1}$, so $\mathbb{R}^n \setminus \{f(0)\} \simeq S^{n-1}$

So, putting things together, $S^{m-1} \simeq \mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{f(0)\} \simeq S^{n-1}$, i.e. $S^{m-1} \simeq S^{n-1}$

By theorem, $H_{m-1}(S^{m-1}) = H_{m-1}(S^{n-1})$

Case 1: $m=1$: $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$, so $n=1$ [prop.]

$\therefore$ So $m=n$

Case 2: $m>1$: $H_{m-1}(S^{m-1}) \cong \mathbb{Z}$

Then $H_{m-1}(S^{n-1}) \cong \mathbb{Z}$ only when $n-1 = m-1$, i.e. $m=n$. $\square$

Problems for Lesson 43: Homotopy Invariance of Homology, II: Sketch of Proof and Applications

April 19, 2017

Problem (1) will be graded.

(1) Prove the **General Brouwer Fixed-Point theorem**: for any $n \geq 1$, show that any map from the n-dimensional closed unit disk D^n to itself has a fixed point.

(2) Suppose $s, t : |K| \to |L|$ are simplicial maps and assume that there are homomorphisms $d_n : C_n(K) - C_{n+1}(L)$ for each n such that

$$d \circ \partial + \partial \circ d = t - s : C_n(K) \to C_n(L).$$

Prove that s and t induce the same homomorphisms on homology groups. These d_n are collectively called a chain homotopy between s and t, which is used in the proof of the homotopy invariance of homology groups.

(3) Recall the definitions of orientability for a closed surface and a combinatorial surface, respectively. We showed that the former implies the latter in L33. Read Theorem 8.15 on Page 191 for the reverse implication. It uses the homotopy invariance of H_2.

4/20 Applications of Homology, I: Degree and the Hairy Ball Theorem

Def: Given a map $f: S^n \to S^n$, we have a homomorphism $\overset{(\text{of } H_n(K))}{} f_*: H_n(\overset{S^n}{K}) \to H_n(K)$ by $[z] \mapsto \lambda[z]$ ($H_n(K) \cong \mathbb{Z}$, z generator). $\lambda = \deg f$ is the degree of f.

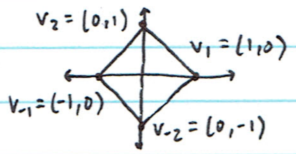
- note: $\deg f$ doesn't depend on the triangulation of S^n
- note: if $f \simeq g$, then $\deg f = \deg g$ (since $f_* = g_*$)
- note: if f is a homeomorphism, then $\deg f = \pm 1$

Pf: Since f homeomorphism, $f_*: H_n(S^n) \to H_n(S^n)$ is an isomorphism. So $f_*([z]) = [z]$ or $[-z]$ □

- note: $\deg \mathrm{id} = 1$
- note: If $f \simeq$ constant map ($S^n \to *$), then $\deg f = 0$.

Def: The triangulation Σ for S^n is defined as follows: In $\mathbb{R}^{n+1}$, let vertices be $v_i = (0, \dots, \overset{i\text{th}}{1}, \dots, 0) \in \mathbb{R}^{n+1}$ and $v_{-i} = (0, \dots, -1, \dots, 0) \in \mathbb{R}^{n+1}$ for $1 \le i \le n+1$.
Simplices in Σ have form $v_{i_1} \cdots v_{i_k}$ where $1 \le |i_1| \le \dots \le |i_k| \le n+1$

Ex: $\mathbb{R}^{1+1} = \mathbb{R}^2 \to 1 \le i \le 2$

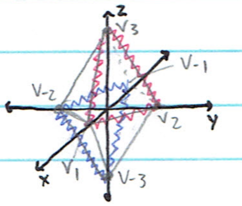


: clearly, radial projection shows us that $|\Sigma| \cong S^1$

Note: A generator $[z] \in H_1(S^1) := H_1(\Sigma) := Z_1(K)/B_1(K) = Z_1(K)$ is of the form
$[z] = [(v_1, v_2) - (v_{-1}, v_2) + (v_{-1}, v_{-2}) - (v_1, v_{-2})]$ (boundary cycle)

- Under the antipodal map $f: S^1 \to S^1$ by $x \mapsto -x$, $f_*([z]) = [(v_{-1}, v_{-2}) - (v_1, v_{-2}) + (v_1, v_2) + (v_{-1}, v_2)]$
  - note that $f_*([z]) = [z] \Rightarrow \deg f = 1$.

Ex: $\mathbb{R}^{2+1} = \mathbb{R}^3 \to 1 \le i \le 3$



: consider $[z] \in H_2(S^2)$
↳ $[z] = [(v_1, v_2, v_3) + \dots - (v_{-1}, v_{-2}, v_{-3}) + \dots]$
Now consider $f: x \mapsto -x$
↳ $f_*([z]) = [(v_{-1}, v_{-2}, v_{-3}) + \dots - (v_1, v_2, v_3) + \dots] = -[z] \Rightarrow \deg f = -1$

$(x \mapsto -x)$

Thm: If $f: S^n \to S^n$ is the antipodal map, then $\deg f = (-1)^{n+1}$

Pf: see previous examples, LTR in book.

Cor: If $f: S^n \to S^n$ has no fixed point, then $\deg f = (-1)^{n+1}$

Pf: Since f has no fixed point, we can define $F: S^n \times I \to S^n$ by

$F(x,t) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$, where F is a well-defined homotopy, from f to $(x \mapsto -x)$

Thus, $\deg f = \deg(x \mapsto -x) = (-1)^{n+1}$ □

Cor: If $f: S^n \to S^n \simeq id: S^n \to S^n$ and n is even, then f has a fixed point.

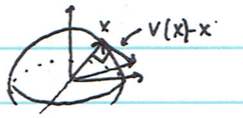
Pf: $\deg f = \deg id = 1$ [previous note]

If f has no fixed point, then $\deg f = (-1)^{n+1} = -1$ ✗ [prev. cor.]

So f has a fixed point. □

Def: A vector field V on S^n is a continuous assignment of tangent vectors to each point of S^n

- Def: A tangent vector is defined as below:



: $v(x)$ is vector from center to x + what we would traditionally think of as the tangent vector.

- note: $v(x) \neq 0$

Def: v is nonvanishing if $\forall x \in S^n$, $v(x) \neq x$ (i.e. $v(x) - x \neq 0$)

Thm: (The Hairy Ball Theorem) If S^n admits a continuous nonvanishing vector field, then n must be odd.

Pf: (By contradiction) Suppose n is even.

Consider cont. nonvanishing vector field $v: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$

Then $f: S^n \to S^n$ by $x \mapsto \frac{v(x)}{|v(x)|}$ is well-defined & continuous [$v(x) \neq 0$]

Then $f \underset{F}{\simeq} id$ by $F: S^n \times I \to S^n$ by $(x,t) \mapsto \frac{(1-t)(v(x)-x) + x}{|(1-t)(v(x)-x) + x|}$

[think: shrinking vector $v(x) - x$ to 0]

Thus, f has a fixed point [cor 2]

So $\exists x \in S^n$ s.t. $\frac{v(x)}{|v(x)|} = f(x) = x \Rightarrow v(x) = x$ ✗ [nonvanishing]

Problems for Lesson 44: Applications of Homology, I: Degree of Maps of Spheres and the Hairy Ball Theorem

April 20, 2017

Problem (1) will be graded.

(1) In this problem, we prove that S^n admits a continuous nonvanishing vector field if and only if n is odd. We do it in two steps.

   (a) Prove that if S^n admits a continuous nonvanishing vector field, then n must be odd. (*Hint:* This was proved in class. You just need to understand it and then reproduce it here.)

   (b) If n is odd, construct a continuous nonvanishing vector field on S^n. (*Hint:* It's in the textbook.)

(2) Prove that if the degree of $f : S^n \to S^n$ is not 1, then f must map some point to its antipode.

(3) If $f : S^n \to S^n$ is a map, and if n is even, show that $f^2 := f \circ f$ must have a fixed point. (*Hint:* Prove that either f has a fixed point, or f sends some point to its antipode. In both cases, f^2 has a fixed point.)

4/21 Applications of Homology, II: The Euler-Poincaré Formula

Recall: Let K be a complex of dimension n. For each $0 \le i \le n$, let α_i be the number of i-simplices in K. Then $\chi(K) := \sum_{i=0}^{n} (-1)^i \alpha_i$

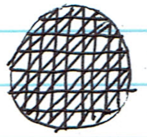
- note that the α_i are the dimensions of $C_i(K)$

Recall: For each i, we know $H_i(K) \cong \mathbb{Z}^{\beta_i} \oplus$ torsion part $((\mathbb{Z}/2\mathbb{Z})^q \oplus \dots)$, where β_i is the i^{th} Betti number.

Thm: (Euler-Poincaré Formula) $\chi(K) = \sum_{i=0}^{n} (-1)^i \beta_i$

- note: Since H_i are homotopy invariant (i.e. H_i only depend on the homotopy type of $|K|$), so are β_i $\Rightarrow$ so is χ
- note: This formula gives an easy way to compute χ

Ex:

$K =$  : $\chi(|K|) = \chi(\cdot) = 1$

can be shrunk to point

Ex: $\chi(\text{torus}) = \beta_0 - \beta_1 + \beta_2 = 1 - 2 + 1 = 0$

Recall: $H_i(K) := Z_i(K)/B_i(K)$, where $B_i(K) < Z_i(K) < C_i(K)$

- the i^{th} chain group $C_i(K) := \{ n_1\sigma_1 + \dots + n_{\alpha_i}\sigma_{\alpha_i} \mid n_j \in \mathbb{Z} \} \cong \mathbb{Z}^{\alpha_i}$

Def: The rational i^{th} chain group of K, $C_i(K, \mathbb{Q}) := \{ n_1\sigma_1 + \dots + n_{\alpha_i}\sigma_{\alpha_i} \mid n_j \in \mathbb{Q} \} \cong \mathbb{Q}^{\alpha_i}$

- note: $C_i(K, \mathbb{Q})$ is a vector space ($C_i(K)$ is not)

defined similarly

Def: The i^{th} homology group with rational coefficients, $H_i(K, \mathbb{Q}) := Z_i(K, \mathbb{Q})/B_i(K, \mathbb{Q})$

- similar definitions for $Z_i(K, \mathbb{Q})$, $B_i(K, \mathbb{Q})$, ∂

Fact: If $H_i(K) \cong \mathbb{Z}^{\beta_i} \oplus$ torsion parts, then $H_i(K, \mathbb{Q}) \cong \mathbb{Q}^{\beta_i}$

Pf: (idea, pf in book) Compare $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Q}/2\mathbb{Q} = \mathbb{Q}/\mathbb{Q} = 0$.

Fact: $C_i(K, \mathbb{Q})$, $B_i(K, \mathbb{Q})$, $Z_i(K, \mathbb{Q})$, $H_i(K, \mathbb{Q})$ are all vector spaces.

Thm: (Euler-Poincaré Formula)

Pf: Consider the chain complex:

$$0 \xrightarrow{\partial} C_n(K,\mathbb{Q}) \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$ [write $C_i(K,\mathbb{Q})$ as C_i]

Since $B_n = 0$, $Z_n = H_n \cong \mathbb{Q}^{\beta_n}$

(labels, not exponents)

Let $z_1^n, \dots, z_{\beta_n}^n$ be a basis for Z_n

Since $Z_n < C_n$, we can add $c_1^n, \dots, c_{\gamma_n}^n \in C_n$ s.t. $z_1^n, \dots, z_{\beta_n}^n, c_1^n, \dots, c_{\gamma_n}^n$ forms a basis for C_n.

So $\boxed{\alpha_n = \beta_n + \gamma_n.}$

Now, span$\{\partial z_1^n, \dots, \partial z_{\beta_n}^n, \partial c_1^n, \dots, \partial c_{\gamma_n}^n\} =: B_{n-1}$

Since $z_i^n \in Z_n$, $\partial z_i^n = 0$.

So B_{n-1} is actually spanned by $\partial c_1^n, \dots, \partial c_{\gamma_n}^n$

$\partial c_1^n, \dots, \partial c_{\gamma_n}^n$ form a basis of B_{n-1}

Pf: They span B_{n-1} [by definition]

Let $a_1 \partial c_1^n + \dots + a_{\gamma_n} c_{\gamma_n}^n = 0$, where $a_i \in \mathbb{Q}$

Then $\partial(a_1 c_1^n + \dots + a_{\gamma_n} c_{\gamma_n}^n) = 0$

So $a_1 c_1^n + \dots + a_{\gamma_n} c_{\gamma_n}^n \in Z_n$ [by definition]

Since Z_n has basis $z_1^n, \dots, z_{\beta_n}^n$, $\exists\, b_1, \dots, b_{\beta_n} \in \mathbb{Q}$ s.t.

$a_1 c_1^n + \dots + a_{\gamma_n} c_{\gamma_n}^n = b_1 z_1^n + \dots + b_{\beta_n} z_{\beta_n}^n \Rightarrow (a_1 c_1^n + \dots) - (b_1 z_1^n + \dots) = 0.$

Since $c_1^n, \dots, c_{\gamma_n}^n, z_1^n, \dots, z_{\beta_n}^n$ form a basis for C_n, we know

$a_1 = \dots = a_{\gamma_n} = b_1 = \dots = b_{\beta_n} = 0$ (c_i^n lin. indep. ✓) □

Since $H_{n-1} := Z_{n-1}/B_{n-1}$, $Z_{n-1} = H_{n-1} \oplus B_{n-1}$ [these are vector spaces]

Choose a basis $z_1^{n-1}, \dots, z_{\beta_{n-1}}^{n-1}$ for H_{n-1}.

Now we can extend to a basis of $C_{n-1} = Z_{n-1} \oplus$ the rest (add $c_1^{n-1}, \dots, c_{\gamma_{n-1}}^{n-1}$)

Thus, $\boxed{\alpha_{n-1} = \gamma_n + \beta_{n-1} + \gamma_{n-1}.}$

$C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$

$H_0 = Z_0/B_0$

$= C_0/B_0$

$C_0 = Z_0 = B_0 \oplus H_0$

$\dim \gamma_1$, $\dim \beta_0$

So: $\alpha_n = \beta_n + \gamma_n$ $\quad$ $(-1)^n \alpha_n = (-1)^n \beta_n + (-1)^n \gamma_n$

$\alpha_{n-1} = \gamma_n + \beta_{n-1} + \gamma_{n-1} \Rightarrow (-1)^{n-1}\alpha_{n-1} = (-1)^{n-1}\gamma_n + (-1)^{n-1}\beta_{n-1} + (-1)^{n-1}\gamma_{n-1}$

...

$\alpha_0 = \gamma_1 + \beta_0$ $\quad$ $\alpha_0 = \gamma_1 + \beta_1$

$\Rightarrow$ add all terms: $\underset{\chi}{\Sigma (-1)^i \alpha_i} = \Sigma (-1)^i \beta_i$ □

Problems for Lesson 45: Applications of Homology, II: The Euler-Poincaré Formula

April 21, 2017

Problem (1) will be graded.

(1) Use the Euler-Poincaré Formula to compute the Euler characteristics of the following spaces.

  (a) mT^2
  (b) $n\mathbb{R}P^2$
  (c) space obtained from mT^2 by removing the interior of r disjoint closed discs
  (d) space obtained from $n\mathbb{R}P^2$ by removing the interior of r disjoint closed discs
  (e) Δ^{100}
  (f) solid torus whose triangulation has a trillion 3-simplices

(2) Understand the proof of the Euler-Poincaré Formula.

MATH 455 Quiz #12 **Name:**______________________

(1) (2 points) True or False? Any map from D^5 to D^5 has a fixed point, where D^5 is the closed unit ball in $\mathbb{R}^5$.

(2) (2 points) Ture or False? Homotopic maps between spaces induce the same homomorphism on homology.

(3) (2 points) True or False? S^3 admits a continuous nowhere vanishing vector field.

(4) (2 points) True or False? If $f : S^n \to S^n$ has no fixed-point, then $\deg f = (-1)^{n+1}$.

(5) (2 points) Ture or False? The Euler characteristic of D^5 is 2.

Applications of Homology, III: The Lefschetz Fixed-Point Theorem

Review: (Linear Algebra)

- Def: The trace of a matrix A, $\mathrm{tr}A$, is $\Sigma_{i=1}^{n} A_{ii}$
- Prop: Given linear operator $T: \mathbb{Q}^n \to \mathbb{Q}^n$, pick basis $b_1, \ldots, b_n$ for $\mathbb{Q}^n$. Let A be the matrix of T w.r.t. this basis. Then $A = [a_{ij}]$ where $T(b_j) = \Sigma_{i=1}^{n} a_{ij} b_i$
  - So linear operator composition 'is' matrix multiplication
  - $\mathrm{tr}(T) := \mathrm{tr}(A)$

↳ $\mathrm{tr}(T)$ doesn't depend on the basis chosen

Pf: Let $c_1, \ldots, c_n$ be another basis for $\mathbb{Q}^n$, $c_j = \Sigma s_{ij} b_i$, let $S = [s_{ij}]$. Then w.r.t. the new basis, T corresponds to the matrix $S^{-1}AS$ [LTR]. $\mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(SS^{-1}A) = \mathrm{Tr}(A)$ [property of trace] $\square$.

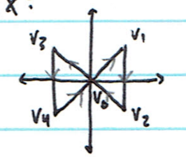
Recall: Given $f: X \to X$. Choose triangulation K s.t. $|K| \cong X$. Then f induces a homomorphism $f_*^i : H_i(K, \mathbb{Q}) \to H_i(K, \mathbb{Q})$ where $H_i(K, \mathbb{Q}) \cong \mathbb{Q}^{\beta_i}$

- note: f_*^i is a linear operator.

Def: The Lefschetz number of f, $\Lambda_f := \Sigma_{i=0}^{n} (-1)^i \mathrm{tr}(f_*^i)$

- fact: Λ_f doesn't depend on the triangulation
- fact: $f \simeq g \Rightarrow \Lambda_f = \Lambda_g$ [since $f_*^i = g_*^i$] dimension of $\left[\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}\right]$
- fact: $\Lambda_{id} = \Sigma(-1)^i \mathrm{tr}\left[\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}\right] = \Sigma(-1)^i \beta_i = \chi(X)$

Ex:



$X = |K| \cong S^1 \vee S^1$ (one-point union of 2 circles)

Define $f: S^1 \vee S^1 \to S^1 \vee S^1$ by reflection w.r.t. y-axis

- $f_*^0 : H_0(K, \mathbb{Q}) \to H_0(K, \mathbb{Q})$ takes $[v_0] \mapsto [v_0]$

↳ matrix for f_*^0: $[1] \to \mathrm{tr} f_*^0 = \mathrm{tr}[1] = 1$.

- $f_*^1 : H_1(K, \mathbb{Q}) \to H_1(K, \mathbb{Q})$ takes $[\underbrace{(v_0, v_1) + (v_1, v_2) + (v_2, v_0)}_{z_1}] \mapsto [\underbrace{(v_0, v_3) + (v_3, v_4) + (v_4, v_0)}_{z_2}]$
  - so $[z_1] \mapsto [z_2]$, $[z_2] \mapsto [z_1]$

↳ matrix for f_*^1: $\begin{array}{cc} \left[\begin{array}{cc} 0 & 1 \\ \bullet & 0 \end{array}\right] & \begin{array}{c} [z_1] \\ [z_2] \end{array} \\ \begin{array}{cc} [z_1] & [z_2] \end{array} & \end{array} \to \mathrm{tr} f_*^1 = \mathrm{tr}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 + 0 = 0$.

- f_*^i for $i \geq 2 \to 0$.

$\Longrightarrow$ So $\Lambda_f := \mathrm{tr} f_*^0 - \mathrm{tr} f_*^1 = 1 - 0 = 1$.

Thm: (Lefschetz Fixed-Point Theorem) If $\Lambda_f \neq 0$, then f has a fixed point

- note: f in previous example has fixed point: $f(v_0) = v_0$
- note: converse does not hold.
  - ex: $id : S^1 \to S^1$ has many fixed points, but $\Lambda_{id} = \chi(S^1) = 0$.

Pf: (sketch of idea, LTR in book)

Consider the contrapositive: f has no fixed point $\Rightarrow \Lambda_f = 0$.

Then:

① f can be deformed to a simplicial map so that $f^i : C_i(K, \mathbb{Q}) \to C_i(K, \mathbb{Q})$ satisfies $f^i(\sigma) \neq \sigma$.

② Thm: (Hopf-trace Theorem) $\sum_{i=0}^{n} (-1)^i \operatorname{tr} f^i = \sum_{i=0}^{n} (-1)^i \operatorname{tr} f_*^i$

Pf: (LTR, similar to proof of Euler-Poincaré Theorem, in book).

So $\operatorname{tr} f^i = \operatorname{tr} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} = 0$ [① → $f^i(\sigma) \neq \sigma$, ② → link to Lefschetz #] □.

$D^n \to D^n$ too.

Ex: Any $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ has a fixed point

Pf: $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$ $\qquad$ $H_0(\mathbb{R}P^2, \mathbb{Q}) \cong \mathbb{Q}$

$H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} \Longrightarrow H_1(\mathbb{R}P^2, \mathbb{Q}) \cong 0$

$H_i(\mathbb{R}P^2) \cong 0 \ (i \geq 2)$ $\qquad$ $H_i(\mathbb{R}P^2, \mathbb{Q}) \cong 0 \ (i \geq 2)$

So $f_*^0 : [v] \mapsto [v]$, so $\operatorname{tr} f_*^0 = 1$.

So $\Lambda_f = 1 - 0 + 0 - \ldots = 1 \neq 0 \Rightarrow f$ has fixed point. □

$f_*^n : \mathbb{Q}^1 \to \mathbb{Q}^1$

Ex: Let $f : S^n \to S^n$. Then $\Lambda_f = 1 + 0 + \ldots + 0 + (-1)^n \operatorname{tr} f_*^n = 1 + (-1)^n \deg f$.

- So if $\deg f : S^n \to S^n \neq (-1)^{n+1}$, then $\Lambda_f \neq 1 + (-1)^n (-1)^{n+1} = 0$

↳ So f has a fixed point.

Problems for Lesson 46: Applications of Homology, III: The Lefschetz Fixed-Point Theorem

April 24, 2017

No problems are to be collected.

(1) Let A and B be two n by n matrices of real (complex, rational, or integral) numbers. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(2) Prove the Hopf Trace Theorem: if $\phi^i : C_i(K, \mathbb{Q}) \to C_i(K, \mathbb{Q})$ form a chain map where K is a complex of dimension n, then

$$\sum_{i=0}^{n} (-1)^i \mathrm{tr}(\phi^i) = \sum_{i=0}^{n} (-1)^i \mathrm{tr}(\phi_*^i),$$

where ϕ_*^i are the induced maps on homology.

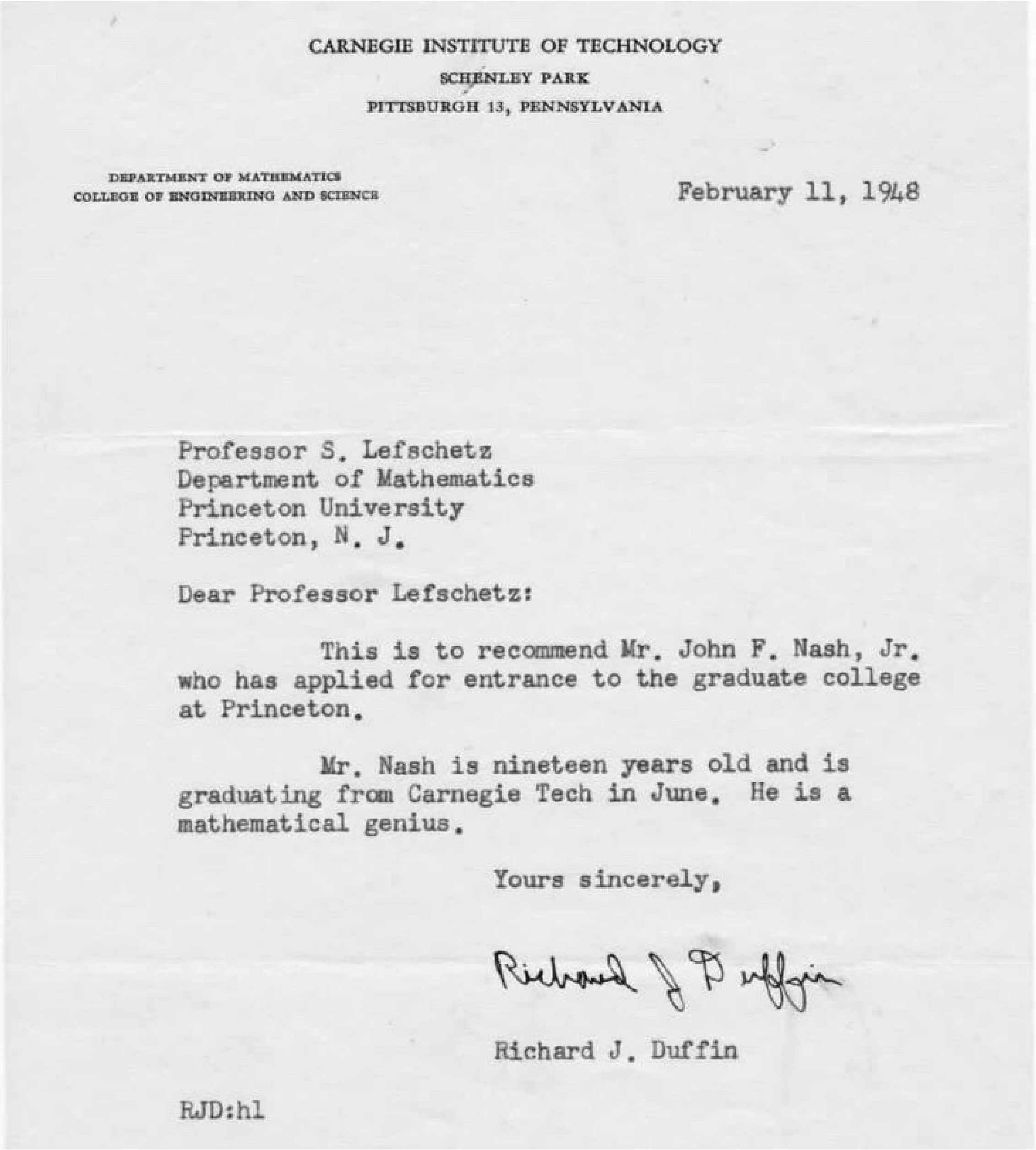
(3) Prove that the Euler characteristic of a compact, path-connected, triangulable topological group must be zero. (In particular, this shows that all even dimensional spheres (which surely are compact, path-connected and triangulable) are not topological groups.)

Hint: Show the following.

(a) If the identity map of X is homotopic to a map which does not have a fixed point, then $\chi(X) = 0$.

(b) If G is a path-connected topological group, then left translation $L_g : G \to G$ defined by $L_g(x) = gx$ is homotopic to the identity. (Let $\gamma : I \to G$ be a path from g to e. Then $H(x, t) = \gamma(t)x$ is a homotopy from l_g to *id*.)

(4) On the next page is a recommendation letter for John Nash's graduate school application from Prof. Duffin to Prof. Lefschetz. Coincidently, both Lefschetz and Nash were involved in fixed-point theories. Duffin's other student Raoul Bott is one of the greatest topologists, though their joint work was in electrical engineering. Bott's students Stephen Smale and Daniel Quillen both got the Fields medal.



CARNEGIE INSTITUTE OF TECHNOLOGY
SCHENLEY PARK
PITTSBURGH 13, PENNSYLVANIA

DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING AND SCIENCE

February 11, 1948

Professor S. Lefschetz
Department of Mathematics
Princeton University
Princeton, N. J.

Dear Professor Lefschetz:

This is to recommend Mr. John F. Nash, Jr. who has applied for entrance to the graduate college at Princeton.

Mr. Nash is nineteen years old and is graduating from Carnegie Tech in June. He is a mathematical genius.

Yours sincerely,

Richard J Duffin

Richard J. Duffin

RJD:hl

(Ch. 10)
L47

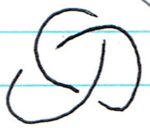
Knots

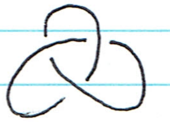
Def: A knot is an embedding $k : S^1 \hookrightarrow \mathbb{R}^3$

- so k is a homeomorphism from S^1 onto its image $k(S^1)$, and $k(S^1)$ has the subspace topology from $\mathbb{R}^3$
- note: we often just call $k(S^1)$ the knot.

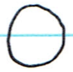
Ex: (left-handed trefoil knot)   Ex: (right-handed trefoil knot)

3 "intersections"



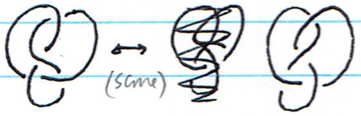


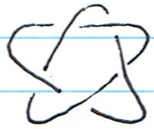
Ex: (Unknot)   Ex: (not a knot)



: Note: $S^1 \to \mathbb{R}^3$ is not an embedding, since it's not an injection (see self-intersection)

Ex: (the figure-8 knot)   Ex: (5 intersections)



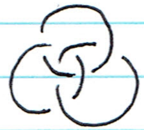


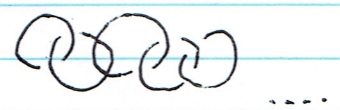
Note: we only consider tame knots, knots w/ finitely many crossings.

disjoint union

Def: A link is an embedding $\ell : \underbrace{S^1 \sqcup S^1 \sqcup \ldots \sqcup S^1}_{\text{finitely many}} \hookrightarrow \mathbb{R}^3$

Ex:





Note: we will mostly consider knots

When are two knots considred the same?

- intuition: if one knot can be deformed continuously into the other w/o self-intersections at each stage, they 'should' be the same knot.

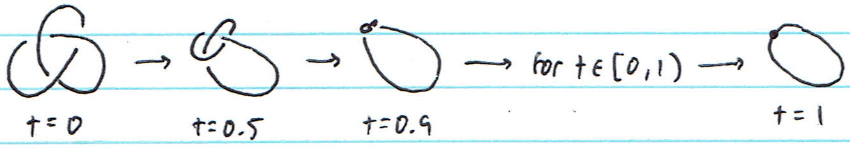
Def: Knots $k_1 : S^1 \hookrightarrow \mathbb{R}^3$ and $k_2 : S^1 \hookrightarrow \mathbb{R}^3$ are equivalent in $\exists$ an isotopy $k_1 \to k_2$.

in general: $F : X \times I \to Y$. ←

Def: An isotopy is a homotopy $F : S^1 \times I \to \mathbb{R}^3$ s.t. $F(x,0) = k_1(x)$, $F(x,1) = k_2(x)$, and for each $t \in I$, $F(x,t) : S^1 \hookrightarrow \mathbb{R}^3$ is an embedding.

Note: this definition is useless, since it makes all knots equivalent!

- ex:



↳ mathematically, this is an isotopy (for all knots)

Def: Knots $k_1, k_2 : S^1 \hookrightarrow \mathbb{R}^3$ are equivalent if $\exists$ an isotopy $F : \mathbb{R}^3 \times I \to \mathbb{R}^3$ s.t. $F(x,0) = id(x)$ and $F(x,1) : \mathbb{R}^3 \to \mathbb{R}^3$ maps $k_1(S^1)$ homeomorphically onto $k_2(S^1)$

- idea: require that some neighborhood of the knot (e.g. $\mathbb{R}^3$) is also deformed nicely.
- note: we say k_1 is equivalent to k_2 through ambient isotopy to the identity
- note: we get a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$
  - restricts to homeomorphism $f : k_1(S^1) \to k_2(S^1)$
  - gives us homeomorphism $f : \mathbb{R}^3 \setminus k_1(S^1) \to \mathbb{R}^3 \setminus k_2(S^1)$

Cor: If $\pi_1(\mathbb{R}^3 \setminus k_1(S^1)) \ncong \pi_1(\mathbb{R}^3 \setminus k_2(S^1))$, then k_1 and k_2 are not equivalent.

Def: $\pi_1(\mathbb{R}^3 \setminus k_1(S^1))$ is the knot group of k_1.

- note: book 10.2 uses Seifert-van Kampen to compute these
  - issue: end up w/ gps expressed by generators, which are difficult to tell apart.

Problems for Lesson 47: Knots – What should be the correct definition of equivalence of knots?

April 26, 2017

No problems are to be collected.

(1) Show that the "left-handed" figure 8 knot is equivalent to the "right-handed" figure 8 knot through ambient isotopy to the identity. You are allowed to prove it using mechanical engineering. An item in the Beginning Topologist's Toolbox is useful.

(2) Can the left-handed trefoil knot be deformed to the right-handed trefoil knot through ambient isotopy to the identity? We will answer this on Friday.

(3) Read Section 10.2. Compute the knot groups for several familiar knots. In particular, compute them for the left- and right- handed trefoils. Are they isomorphic?

(4) Look up Tait's theory of the periodic table of elements using knots.

(5) Given two knots k_1 and k_2, we can take their connected sum $k_1 \# k_2$. You can google what that means. If a knot k can not be written as a connected sum $k_1 \# k_2$ where neither k_1 nor k_2 is the unknot, then we say k is called a prime knot. All the unknot we saw in class today are prime knots. Any knot table you encounter is also likely a table of only prime knots. Download such a knot table. Skim through it. Can you find some knots in it which have cultural meanings?

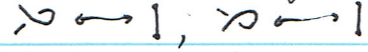
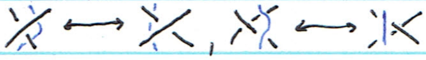
How do we distinguish knots? Jones Polynomials

Motivation: recall $\pi_1(\mathbb{R}^3 \setminus k(S^1))$ using Seifert-van Kampen thm ~~is~~ are written in terms of generators & relations, so they're hard to tell apart.

Idea: If $k_1 \sim k_2$ (thru ambient isotopy to the identity), then $J(k_1) = J(k_2)$

- i.e. Jones Polynomials are knot invariant
- so if $J(k_1) \neq J(k_2)$, then $k_1 \nsim k_2$.

Note: we're actually studying 'planar projections' of knots. So we can 'stretch the paper,' ~~and~~ or do basic moves to the planar diagram w/o changing the knot:

- Reidemeister move of Type I: 
- Reidemeister move of Type II: 
- Reidemeister move of Type III: 

Thm: (Reidemeister) If $k_1 \sim k_2$, then we can obtain k_2 from k_1 (or v.v.) by a finite number of Reidemeister moves. (identifying k w/ its planar diagram)

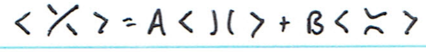
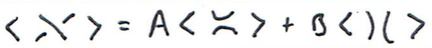
We will define Jones Polynomials (thru Louis Kauffman's approach):

① Define bracket polynomial $\langle K \rangle$

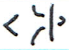
② Define $[K]$

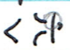
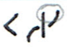
③ Define $J(K)$ (in literature, $V(t)$)

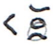
We set some rules:

- Rule 1: $\langle \bigcirc \rangle = 1$ [unknot]
- Rule 2:  [A, B formal variables]
  - note: 
- Rule 3: $\langle K \cup \bigcirc \rangle = C \langle K \rangle$ [not linked to unknot]

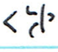
Recall: We want $\langle\ \rangle$ to be invariant under the moves

- Type II: want $\langle$  $\rangle = \langle\,|\,|\,\rangle$

↳ $\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$ [rule 2]

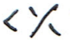
$= A(A\langle \asymp \rangle + B\langle)(\rangle) + B(A\langle$  $\rangle + B\langle$  $\rangle)$ [rule 2]

$= (A^2+ABC+B^2)\langle \asymp \rangle + AB\langle)(\rangle$ [rule 3, distribution]

- note: $\langle$  $\rangle = \langle)(\rangle$, so we can match coefficients

↳ $A^2+ABC+B^2 = 0,\ AB = 1$: so $B = A^{-1}$, $C = -A^2-A^{-2}$

$\Longrightarrow$ Rule 1: $\langle O \rangle = 1$

Rule 2: $\langle$  $\rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

Rule 3: $\langle K \cup O \rangle = (-A^2-A^{-2})\langle K \rangle$

- note: this is also invariant for type III moves

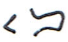
<u>Pf</u>: $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ [rule 2]

$= A\langle)|(\rangle + A^{-1}\langle$  $\rangle$ [type II, planar isotopy]

$= A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ [type II]

$= \langle$  $\rangle$ $\square$

- note: this is not invariant for type I moves

<u>Pf</u>: $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle)O\rangle$ [rule 2]

$= A\langle|\rangle + A^{-1}(-A^2-A^{-2})\langle|\rangle$ [planar isotopy, rule 3]

$= -A^{-3}\langle|\rangle$

- similarly, $\langle$  $\rangle = -A^3\langle|\rangle$

Problems for Lesson 48: How do we distinguish the first few knots? – The Jones Polynomial

April 27, 2017

No problems are to be collected.

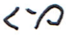
(1) Read the Scientific American article *Knot Theory and Statistical Mechanics* by Vaughan F. R. Jones himself. It's posted in Moodle.

(2) Read the fresh article (as fresh as today's breakfast because it just came out this morning) about *Virtual Knot* invented by Louis Kauffman following ideas of Gauss. It has many connections to classical knot theory (the knot theory we are studying) and many other areas of mathematics. This article also has a good overview of that much knot theory we have talked about.

http://www.ams.org/publications/journals/notices/201705/rnoti-p461.pdf

The End.

Recall: $\langle\ \rangle$ is only invariant w.r.t. Reidemeister moves of type II & III

- $\langle$  $\rangle = -A^{-3}\langle 1 \rangle$, and $\langle$  $\rangle = -A^{3}\langle 1 \rangle$

Def: Given a knot, label a direction. At each crossing, there are two possibilities:  - assign $+1$,  - assign -1. The writhe number of a knot k, $w(k) := \sum_{\text{crossing \#s}}$

- ex:  : $w(\text{left-handed trefoil}) = -1 + -1 + -1 = -3$

- note: w is independent of the chosen direction
- note: w is invariant under R. II & III

Pf: Type II : $w($  $) = 1 + -1 = 0 = w(11)$

Type III : $w($  $) = 1 + 1 - 1 = 1 = w($  $)$

- note: for R. I moves, we add/subtract 1
  - ex: $w($  $) = -1 \xrightarrow{+1} w(1) = 0.$

Def: $[K] = (-A^3)^{-w(K)}\langle K \rangle$

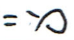
Thm: $[\ ]$ is a knot invariant.

Pf: Both w & $\langle\ \rangle$ are invariant under R. II & III, so $[\ ]$ is too.

Let $K =$ , $K' = 1$

Then $[K] = (-A^3)^{-w(K)}\langle K \rangle = (-A^3)^{-(w(K')-1)}(-A^{-3})\langle K' \rangle$

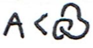
$= (-A^3)^{-w(K')}(-A^3)(-A^{-3})\langle K' \rangle = (-A^3)^{-w(K')}\langle K' \rangle = [K']$

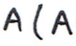
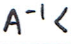
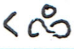
Similar for $K =$  $\square$.

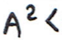
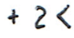
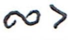
Def: The Jones Polynomial of K, $J(K) := [K]$ where $A = t^{-\frac{1}{4}}$

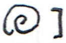
- note: the exponents will always end up as integers.

Ex: Jones polynomial of right-handed trefoil is $-t^4+t^3+t$

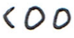
Pf: $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ [rule 2]

$= A(A\langle$  $\rangle + A^{-1}\langle$  $\rangle) + A^{-1}(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$ [rule 2]

$= A^2\langle$  $\rangle + 2\langle$  $\rangle + A^{-2}(-A^2-A^{-2})\langle$  $\rangle$ [rule 3]

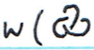
$= (1-A^{-4})\langle$  $\rangle + A^2(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$ [rule 2 on ]

$= (1-A^{-4})\langle$  $\rangle + A^2(A(-A^2-A^{-2})+A^{-1})$ [rule 1]

$= (1-A^{-4})(A\langle$  $\rangle + A^{-1}\langle$  $\rangle) - A^5$ [rule 2]

$= (1-A^{-4})(A+A^{-1}(-A^2-A^{-2})) - A^5$ [rule 1, 3]

$= A^{-7} - A^{-3} - A^5$

$w($  $) = 1+1+1 = 3$

$[$  $] = (-A^3)^{-3}(A^{-7}-A^{-3}-A^5)$

$= -A^{-16} + A^{-12} + A^{-4}$

So $J($  $) = -t^4+t^3+t$ □

Note: Jones polynomials can't distinguish all knots, but it's very useful.

Computations of Jones Polynomials

Rules:

① $\langle O \rangle = 1$

② $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

③ $\langle K \cup O \rangle = (-A^2-A^{-2})\langle K \rangle$

Afterwards,

$[K] = (-A^3)^{-w(K)}\langle K \rangle$

where $w(K)$ is the writhe number of K, which is computed by orienting K but w doesn't depend on the orientation.

Finally, replace A by $t^{-\frac{1}{4}}$, we get the Jones polynomial $J(K)$.

Example ① J().

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= A(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$

$+ A^{-1}(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$

$= A^2\langle$  $\rangle + \langle$  $\rangle + \langle$  $\rangle$

$+ A^{-2}\langle$  $\rangle$

$= A^2(-A^2-A^{-2})\langle$  $\rangle + 2\langle$  $\rangle$

$+ A^{-2}(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$

$= (-A^4-1+2)\langle$  $\rangle + A^{-1}$

$+ A^{-3}(-A^2-A^{-2})\langle O \rangle$

$= (-A^4+1)(A\langle OO \rangle + A^{-1}\langle O \rangle) + A^{-1} - A^{-1} - A^{-1}$

$= (-A^4+1)(A(-A^2-A^{-2})\langle O \rangle + A^{-1}) - A^{-5}$

$= (-A^4+1)(-A^3-A^{-1}+A^{-1}) - A^{-5}$

$= A^7 - A^3 - A^{-5}$

So $[$  $] = (-A^3)^{-w(}$  $^{)}\langle$  $\rangle$

$= -A^9(A^7-A^3-A^{-5})$

$= -A^{16}+A^{12}+A^4$

Thus, J() $= -t^{-4}+t^{-3}+t^{-1}$.

Exercise:

② J() $= t + t^3 - t^4$.

So,  and  are not equivalence ~~through~~ ambient isotopy to the ~~identity~~.

③ J() $= t^{-2}-t^{-1}+1-t+t^2$

④ J() $= -t^{-7}+t^{-6}-t^{-5}+t^{-4}+t^{-2}$

⑤ J() $= t^2+t^4-t^5+t^6-t^7$

⑥ J() $= -t^{-6}+t^{-5}-t^{-4}+2t^{-3}-t^{-2}+t^{-}$

⑦ J() $= t-t^2+2t^3-t^4+t^5-t^6$

The above are all the prime knots up to five crossings.

J() = J()

$= (t^{-2}-t^{-1}+1-t+t^2)^2$

and the two knots are not equivalen

Problems for Lesson 49: The End

April 28, 2017

No problems are to be collected.

(1) Compute the Jones polynomial for all the prime knots up to five crossings. Answers are in the handout.

Final Exam Study Guide

The final exam will take place on **Monday, May 8th**, in **Seeley Mudd 204** from **9:00 A.M. to 11:59 A.M.** It covers everything we learned this semester. You will not be allowed to use notes, books, calculators, etc. All you need are pencils (pens) and erasers.

The exam will have 9 problems. The total number of points is 100. Each problem may have several parts. You may be asked to state a definition, state a theorem, judge whether a statement is true or false, or prove a statement. If you are asked for a proof, you have to give a logically correct proof written in English sentences. Scratch work is not considered a proof.

Exam problems will be similar to quiz problems, homework problems and anything we did in class. Carefully go through your notes and homework.

A practice exam has been posted in Moodle. Treat that as a real exam. Find a nice and quiet place and then try it within the 3-hour time constraint. (You don't really need that much time.) The **solution** is also posted in Moodle so that you know what I expect from you. **Compared to the midterms, the final exam contains more variations and one or two problems you have never seen.**

On the Friday (May 5) of the reading period, I will answer your questions in an optional review session. **SMUD 207** has been reserved from **11:00 to 11:59 A.M.** for it.

Below is a list of topics from L37 to L49 we covered after Exam 2.

- L37: Homology: Intuitive Ideas and Introductory Examples
    - Why do we need more algebraic topology?
    - Poincaré's idea of associating a number (Betti) to each dimension i where i is the number of $(i+1)$-dimensional cavities bounded by an i-dimensional closed surface with singularity
    - Noether's idea of associating an abelian group to each dimension i
    - lots of examples

- L38: Homology: Definition and First Computations
    - oriented simplex
    - definition of chain groups
    - definition of boundary homomorphisms ∂
    - $\partial^2 = 0$
    - definition of the cycle group
    - definition of the boundary group
    - definition of the homology group
    - definition of the Betti numbers
    - definition of homologous cycles
    - compute the homologies of the boundary of a triangle by hand
    - compute the homologies of the boundary of a square by hand
    - compute the homologies of the boundary of a tetrahedron by hand

- L39: Homology of Cones and thus of Spheres
    - compute the homology of the solid triangle by hand
    - compute the homologies of the solid tetrahedron by hand
    - computation of $H_0(K)$ where K is any complex
    - homology of cones and its proof
    - homology of spheres from the homology of simplices as cones

- L40: Homology of Surfaces
    - computation of $H_0(K)$ where K is path-connected
    - computation of $H_1(K)$ by abelianizing $\pi_1(|K|)$ where K is any complex

  - sketch the proof of the above
  - computation of $H_2(K)$ where K is a closed surface
  - computation of $H_2(K)$ where K is a compact surface with boundary

- L41: Chain Maps between Chain Complexes
  - definition of chain complex
  - definition of chain map
  - prove that chain map induces homomorphism on homology
  - prove that $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ where ψ and ϕ are chain maps
  - chain map induced from simplicial map

- L42: Homotopy Invariance of Homology, I: Barycentric Subdivision is a sequence of Stellar Subdivisions
  - definition of stellar subdivision
  - express a barycentric subdivision as a sequence of stellar subdivisions
  - (*) stellar subdivision doesn't change the homology of a complex
  - so iterated barycentric subdivision doesn't change the homology of a complex
  - subdivision chain map χ
  - the standard simplicial map θ
  - the usage of the above two maps in proving (*)

- L43: Homotopy Invariance of Homology, II: Sketch of Proof and Applications
  - the three facts about homomorphism on homology induced from map between triangulable spaces
  - proof of the homotopy invariance of homology from the above three facts
  - homologies of S^n for all $n \geq 0$
  - proof that $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$.
  - proof of the general Brouwer fixed-point theorem

- L44: Applications of Homology, I: Degree of Maps of Spheres and the Hairy Ball Theorem
  - definition of degree of a map from a sphere to itself
  - properties of degree
  - degree of the antipodal map
  - the proof that if $f : S^n \to S^n$ doesn't have a fixed-point, then $\deg f = (-1)^{n+1}$
  - the proof that if $f : S^n \to S^n$ is homotopic to the identity and n is even, then f has a fixed point
  - definition of vector fields
  - the hairy ball theorem and its proof
  - more applications of degrees

- L45: Applications of Homology, II: The Euler-Poincaré Formula
  - recall the definition of Euler characteristic
  - The Euler-Poicaré formula and its significance
  - homology with rational coefficients and its relation to homology with integer coefficients
  - Proof of the Euler-Poincaré Formula (its essentially a linear algebra problem)
  - applications

- L46: Applications of Homology, III: The Lefschetz Fixed-Point Theorem
  - trace of a square matrix
  - trace of a linear transformation (operator) and why it's well-defined
  - definition of Lefschetz number and example of its computation
  - The Hopf Trace Theorem
  - The Leftschetz-fixed point theorem and the sketch of its proof
  - applications of the theorem to balls, real projective planes and spheres etc.

  - A path-connected compact triangulable topological group has Euler characteristics 0.

- L47: Knots – What should be the correct definition of equivalence of knots?
  - definition of knots, links and lots of examples
  - the problem of defining equivalence of knots using isotopy only
  - the definition of equivalence of knots using ambient isotopy to the identity
  - connected sum of knots
  - the definition of knot group

- L48: How do we distinguish knots? – The Jones Polynomial
  - the problem with knot group
  - plane isotopy
  - Reidemeister moves of type I, II and III
  - the bracket polynomial $\langle K \rangle$ of a knot and its invariance under moves of type II and III
  - the problem of the bracket polynomial under move of type I

- L49: The End
  - writhe number and its independence of orientation
  - definition of Jones polynomial and the proof that it's a knot invariant
  - computations of Jones polynomial
  - so the unknot, the left trefoil, the right trefoil, the figure eight etc. are all distinct knots

Math 455 Topology, Spring 2017
Practice Final Exam
May 8

Name:

1. (20 points) For the following problems, just write T or F.

   (a) (2 points) The left-handed figure-8 knot and the right-handed figure-8 knot are equivalent through an ambient isotopy to the identity.

   (b) (2 points) The left-handed trefoil knot and the unknot are not equivalent through an ambient isotopy to the identity.

   (c) (2 points) There is a continuous nowhere vanishing vector field on S^3.

   (d) (2 points) If A, B, C are path-connected, then so is $A \times B \times C$.

   (e) (2 points) $\mathbb{R}P^2$ is compact.

   (f) (2 points) Any connected space is also path-connected.

   (g) (2 points) Let A, B, C, D be spaces. If $A \simeq B$ and $C \simeq D$, then $A \times C \simeq B \times D$.

   (h) (2 points) If K is a four dimensional simplicial complex, then $H_5(X) \cong 0$.

   (i) (2 points) Both the cylinder and the Möbius strip deformation retract to a circle.

   (j) (2 points) The inclusion of S^1 onto the boundary circle of the Möbius strip M^2 induces a homomorphism sending a generator in $\pi_1(S^1)$ to ± of twice of a generator in $\pi_1(M^2)$.

2. (10 points) Prove **the pasting lemma**: Let A and B be closed subsets of the space X and $A \cup B = X$. If $f : A \to Y$ and $g : B \to Y$ are continuous functions and they agree over $A \cap B$, namely $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B$ is also a continuous function.

3. (10 points) Prove that if X is a Hausdorff space and A a compact subset of X, then A is closed in X.

4. (10 points) Prove that $[0,1]/\{0,1\}$ is homeomorphic to S^1.

5. (10 points) Let $\alpha : I \to X$ and $\beta : I \to X$ be two paths in the space $X = \mathbb{R}^2 \backslash \{(0,0)\}$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \text{ and } \beta(s) = (\cos(\pi s), -\sin(\pi s)).$$

Prove that $\alpha \not\simeq \beta$ rel $\{0,1\}$. Justify all your claims.

6. (10 points) State and prove the general Brouwer-fixed point theorem.

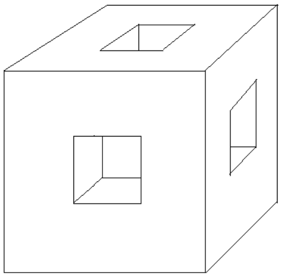
7. (10 points) Let X be the path-connected and compact triangulable space $\mathbb{R}P^2$.

   (a) (4 points) Compute the Euler characteristic $\chi(X)$.

   (b) (6 points) Prove that any map $f : X \to X$ has a fixed-point.

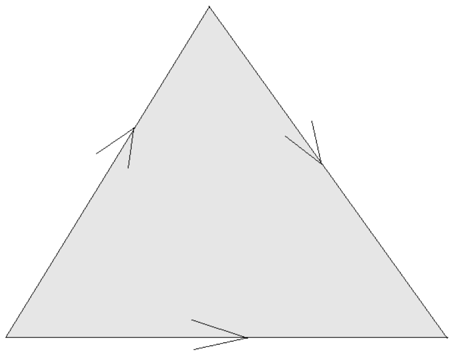
8. (10 points) Two closed surfaces S_1 and S_2 are shown below.

- (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem. (Both are the surfaces you saw in the homework.)
- (3 points) Sketch their polygonal models.
- (4 points) Compute their fundamental groups.


S_1


S_2

9. (10 points) X is the space obtained by identifying all the three edges of a solid triangle (area is filled in) along directions shown below. (X is called the Dunce hat.) Use the Seifert-van Kampen Theorem to compute the fundamental group of X.



Math 455 Topology, Spring 2017
Practice Final Exam
May 8

Name:

Answer Keys

Some of the detailed proofs are referred to class notes, homework or previous tests.

1. (20 points) For the following problems, just write T or F.

   (a) (2 points) The left-handed figure-8 knot and the right-handed figure-8 knot are equivalent through an ambient isotopy to the identity.

   T

   (b) (2 points) The left-handed trefoil knot and the unknot are not equivalent through an ambient isotopy to the identity.

   T

   (c) (2 points) There is a continuous nowhere vanishing vector field on S^3.

   T

   (d) (2 points) If A, B, C are path-connected, then so is $A \times B \times C$.

   T

   (e) (2 points) $\mathbb{R}P^2$ is compact.

   T

   (f) (2 points) Any connected space is also path-connected.

   F

   (g) (2 points) Let A, B, C, D be spaces. If $A \simeq B$ and $C \simeq D$, then $A \times C \simeq B \times D$.

   T

   (h) (2 points) If K is a four dimensional simplicial complex, then $H_5(X) \cong 0$.

   T

   (i) (2 points) Both the cylinder and the Möbius strip deformation retract to a circle.

   T

   (j) (2 points) The inclusion of S^1 onto the boundary circle of the Möbius strip M^2 induces a homomorphism sending a generator in $\pi_1(S^1)$ to ± of twice of a generator in $\pi_1(M^2)$.

   T

This is #1 of the Hw of L13

2. (10 points) Prove **the pasting lemma**: Let A and B be closed subsets of the space X and $A \cup B = X$. If $f : A \to Y$ and $g : B \to Y$ are continuous functions and they agree over $A \cap B$, namely $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B$ is also a continuous function.

Proof: Let C be any closed subset of Y. We will prove h is continuous by showing that $h^{-1}(C)$ is closed in X.

Notice that $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

Since $f: A \to Y$ is continuous, $f^{-1}(C)$ is closed in A.

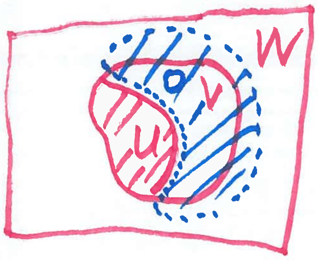
Since $g: B \to Y$ is continuous, $g^{-1}(C)$ is closed in B.

Now we want to show that $f^{-1}(C)$ and $g^{-1}(C)$ are also closed in X so as to conclude that their union, which is $h^{-1}(C)$ is closed in X.

How do we prove it? It follows from the general fact that: If U is closed in V and V is closed in $\underset{\text{total space}}{W}$, then U is also closed in W.

Proof: Since U is closed in V, it means $V \setminus U$ is open in V. By the definition of subspace topology, it means there is an open set O in W such that

$$V \setminus U = O \cap V.$$



Thus, $$U = V \setminus O = \underbrace{\underbrace{V}_{\text{closed in } W} \cap (\underbrace{W \setminus O}_{\text{closed in } W})}_{\text{so closed in } W}$$ □

Since $f^{-1}(C)$ is closed in A and A is closed in X, by the above fact, $f^{-1}(C)$ is closed in X. Similarly, $g^{-1}(C)$ is also closed in X. Therefore, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X. □

3. (10 points) Prove that if X is a Hausdorff space and A a compact subset of X, then A is closed in X.

This is the proof of Theorem 1 of L7.

4. (10 points) Prove that $[0,1]/\{0,1\}$ is homeomorphic to S^1.

This is #4 of Exam 2.

5. (10 points) Let $\alpha : I \to X$ and $\beta : I \to X$ be two paths in the space $X = \mathbb{R}^2 \backslash \{(0,0)\}$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \text{ and } \beta(s) = (\cos(\pi s), -\sin(\pi s)).$$

Prove that $\alpha \not\simeq \beta$ rel $\{0,1\}$. Justify all your claims.

This is #2 of the Hw for L24.

6. (10 points) State and prove the general Brouwer-fixed point theorem.

This is #1 of Hw for L43.

This is an example in class.

7. (10 points) Let X be the path-connected and compact triangulable space $\mathbb{R}P^2$.

(a) (4 points) Compute the Euler characteristic $\chi(X)$.

$H_0(X) \cong \mathbb{Z}$

$H_1(X) \cong \mathbb{Z}/2\mathbb{Z}$

$H_i(X) \cong 0, i \geqslant 2$

So $H_0(X, \mathbb{Q}) \cong \mathbb{Q}^1$

$H_1(X, \mathbb{Q}) \cong 0$

$H_i(X, \mathbb{Q}) \cong 0, i \geqslant 2$.

Thus, by the Euler-Poincaré Formula,

$\chi(X) = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \cdots = 1 - 0 + 0 - 0 \cdots = 1$

(b) (6 points) Prove that any map $f : X \to X$ has a fixed-point.

Proof: Let $f: X \to X$ be any map.

Then f induces maps on homologies:

$f_*^i : H_i(X) \to H_i(X)$.

The Lefschetz number

$$\Lambda_f = \sum_{i=0}^{2} (-1)^i \operatorname{tr} f_*^i$$

$$= \operatorname{tr} f_*^0 - \operatorname{tr} f_*^1 + \operatorname{tr} f_*^2$$

$$= \operatorname{tr}[1] - 0 + 0$$

$$= 1 \neq 0$$

because H₁ and H₂ are zero-dimensional vector spaces

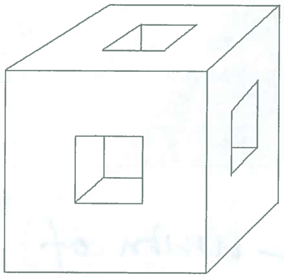
because $f_*^0 : [v] \to [v]$

$\uparrow$ the only generator of H_0

By the Lefschetz fixed-point theorem, $f: X \to X$ must have a fixed-point. $\square$

8. (10 points) Two closed surfaces S_1 and S_2 are shown below.

- (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem. (Both are the surfaces you saw in the homework.)
- (3 points) Sketch their polygonal models.
- (4 points) Compute their fundamental groups.



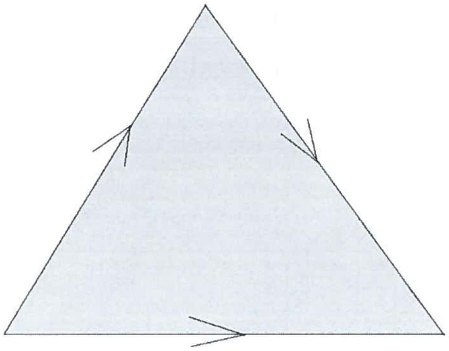
S_1

S_2

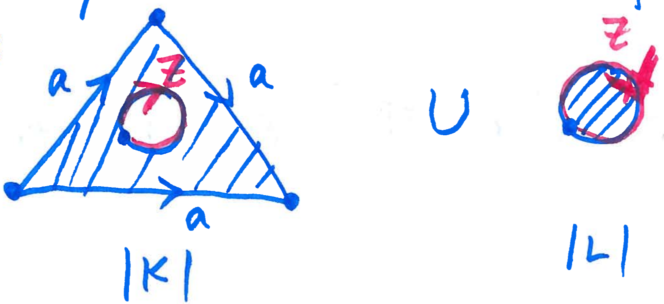
This is Prob #5 of Practice Exam 2.

Dunce hat is introduced in the Hw of L23.

9. (10 points) X is the space obtained by identifying all the three edges of a solid triangle (area is filled in) along directions shown below. (X is called the Dunce hat.) Use the Seifert-van Kampen Theorem to compute the fundamental group of X.



Let's write the space X as the union of $|K|$ and $|L|$ as follows:



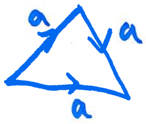
where K and L are some complexes triangulating $|K|$ and $|L|$.

Label the three edges by the common letter a.
(Notice that all three vertices are identified.)
So a represents a circle.

By Seifer-van Kampen:

$\pi_1(X) \cong \pi_1(|K|) * \pi_1(|L|) / \sim$

detail

$|K|$ deformation retracts to  , which is 

So $\pi_1(|K|) \cong \langle a \rangle$

$|L|$ is contractible, so $\pi_1(|L|) \cong \{0\}$.

Let $k: |K \cap L| \hookrightarrow |K|$ and $l: |K \cap L| \hookrightarrow |L|$ be the inclusions.

Let $[z]$ be a generator of $\pi_1 |K \cap L| \cong \mathbb{Z}$ shown in the picture.

Then $k_*([z]) = a \cdot a \cdot a^{-1} = a$ and $l_*([z]) = 0$

So

$\pi_1(X) \cong \langle a \rangle * \{0\} \Big/ a = k_*([z]) = \underset{\overset{\|}{0}}{l_*([z])}$

$\cong 0$

Math 455 Topology, Spring 2017
Final Exam
May 8

Name:

1. (20 points) For the following problems, just write T or F.

   (a) (2 points) There is an isotopy from the left-handed trefoil knot to the unknot.

   (b) (2 points) There is a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ which maps a left-handed trefoil knot to a right-handed trefoil knot.

   (c) (2 points) S^{126} doesn't admit a continuous nowhere vanishing vector field.

   (d) (2 points) The Euler characteristic of $\partial\Delta^4$ is 2.

   (e) (2 points) Let A, B, C be closed surfaces. If $A \ncong B$, then $A\#C \ncong B\#C$.

   (f) (2 points) Let A, B, C be spaces. If $A \ncong B$, then $A \times C \ncong B \times C$.

   (g) (2 points) A space consisting of finitely many points is compact in any topology.

   (h) (2 points) A path-connected space is always connected.

   (i) (2 points) The product of two connected spaces is always connected.

   (j) (2 points) There is a retraction from S^1 to the point $(1, 0) \in S^1$.

2. (8 points) Prove this version of **the pasting lemma**: Let A and B be open subsets of the space X and $A \cup B = X$. If $f : A \to Y$ and $g : B \to Y$ are continuous functions and they agree over $A \cap B$, namely $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B$ is also a continuous function.

3. (12 points)

(a) (6 points) Let X be a Hausdorff space and A a compact subset of X. Prove that A is closed in X.

(b) (6 points) Let X be a Hausdorff space and Y its one-point compactification. Prove that the original topology $\mathcal{T}$ on X and the subspace topology $\mathcal{T}'$ which X inherites from Y are the same.

4. (10 points) Prove that $\mathbb{R}P^1$, defined as the quotient space obtained from S^1 by identifying each pair of antipodal points, is homeomorphic to S^1.

5. (8 points) Let $\alpha : I \to X$ and $\beta : I \to X$ be two paths in the space $X = \mathbb{R}^2 \backslash \{(0,0)\}$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \text{ and } \beta(s) = (\cos(\pi s), -\sin(\pi s)).$$

Prove that with the end points fixed, α cannot be deformed continuously to β in X. More precisely, show that $\alpha \not\simeq \beta$ rel $\{0, 1\}$. Justify all your claims.

6. (7 points) Prove that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$.

7. (10 points) Let X be the path-connected and compact triangulable space $\mathbb{R}P^4$. Its homology groups are as follows.

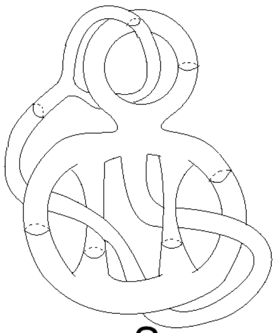
$$H_0(X) \cong \mathbb{Z}, H_1(X) \cong \mathbb{Z}/2\mathbb{Z}, H_2(X) \cong 0, H_3(X) \cong \mathbb{Z}/2\mathbb{Z}, H_i(X) \cong 0, i \geq 4.$$

(a) (2 points) Compute the Euler characteristic $\chi(X)$.

(b) (4 points) Prove that any map $f : X \to X$ has a fixed-point.

(c) (4 points) Can X be a topological group? (Clearly state any theorem you use and prove any statement you make.)

8. (15 points) Two closed surfaces S_1 and S_2 are shown below.



S_1

S_2

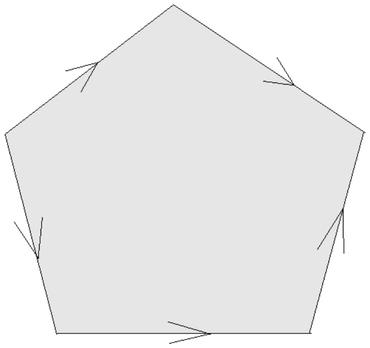
(a) (3 points) Identify each of the two surfaces with a standard surface on the list of the classification theorem.

(b) (3 points) Sketch their polygonal models.

(c) (3 points) Compute their fundamental groups.

(d) (6 points) Let C_1 be a compact surface obtained from S_1 by removing the interiors of 3 disjoint closed discs. Let C_2 be a compact surface obtained from S_2 by removing the interiors of 5 disjoint closed discs. Compute all the homology groups of C_1 and C_2.

9. (10 points) X is the space obtained by identifying all the five edges of a solid pentagon (area is filled in) along directions shown below.



(a) (5 points) Use the Seifert-van Kampen Theorem to compute the fundamental group of X.

(b) (5 points) Is X is contractible? Justify your claim.