Poisson structures from corners of field theories

based on joint work with

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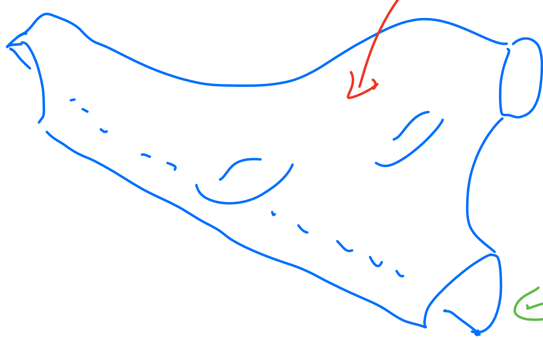
G. Canepa: focus on 3+1 gravity in coframe formulation

Field theory on manifolds with boundaries and corners



bulk: (BV) quantization

$\rightsquigarrow$ partition fn / state ψ

boundary: symplectic manifold M

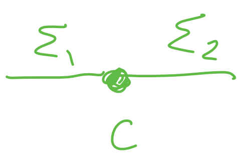
$\rightsquigarrow$ its quantization $\mathcal{H}$

$\cup\!\!\!\shortmid$

ψ

corners: Poisson manifold P and its quantization A

$\underset{\text{boundary}}{\Sigma} \rightsquigarrow M_\Sigma$ | $\rightsquigarrow \mathcal{H}_\Sigma$



$\rightsquigarrow M_{\Sigma_1 \cup \Sigma_2} = M_1 \times_{P_c} M_2$ | $\mathcal{H}_{\Sigma_1 \cup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes_{A_c} \mathcal{H}_{\Sigma_2}$

Part I: Some background & results

Affine Lie algebras

Let $(\mathfrak{g}, \langle , \rangle)$ be a f.d. quadratic Lie algebra (i.e. $\langle , \rangle$ invariant inner product)

- $\mathfrak{g}_{S^1} := \mathrm{Map}(S^1, \mathfrak{g})$ with pointwise Lie bracket
- $c(f, g) := \int_{S^1} \langle f, dg \rangle$ is a cocycle!

$f, g \in \mathfrak{g}_{S^1}$

$\Rightarrow \quad \hat{\mathfrak{g}} := \mathfrak{g}_{S^1} \oplus \mathbb{R}$ with $[f \oplus a, g \oplus b] := [f, g] \oplus c(f, g)$

is a Lie algebra

- $\hat{\mathfrak{g}}$ is called an affine Lie algebra
- Its representation theory is widely studied
- It is related to Chern-Simons theory and the Wess-Zumino-Witten model
- The Poisson algebra in the title is "$\hat{\mathfrak{g}}^*$"

viz., $\mathcal{F}^{\partial\partial} := \Omega^1(S^1)\otimes\mathfrak{g}^*$ with affine Poisson structure (defined on certain functionals, e.g., local ones)

- Associated to the corners of CS theory

2D generalization

Let $\mathfrak{g}$ be a f.d. Lie algebra

Γ a closed oriented surface

$\mathfrak{g}_\Gamma := \mathrm{Map}(\Gamma, \mathfrak{g})$ with pointwise Lie bracket

$\mathfrak{g}^*_\Gamma := \Omega^1(\Gamma) \otimes \mathfrak{g}^*$ with pointwise coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$

$\tilde{\mathfrak{g}}_\Gamma := \mathfrak{g}_\Gamma \oplus \mathfrak{g}^*_\Gamma$ with Lie bracket

$$[\theta \oplus \alpha, \varphi \oplus \beta]^{\sim} := [\theta, \varphi] \oplus (\mathrm{ad}^*_\theta \beta - \mathrm{ad}^*_\varphi \alpha)$$

Cocycle: $\quad c(\theta\oplus\alpha, \varphi\oplus\beta) = \int_\Gamma \big((\alpha, d\varphi) - (\beta, d\theta) \big)$

$(\, , \,)$ canonical pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$

$\Rightarrow \quad \hat{\mathfrak{g}}_\Gamma := \tilde{\mathfrak{g}}_\Gamma \oplus \mathbb{R} \quad$ with

$$[\theta\oplus\alpha\oplus a, \varphi\oplus\beta\oplus b] := [\theta\oplus\alpha, \varphi\oplus\beta]^\sim \oplus c(\theta\oplus\alpha, \varphi\oplus\beta)$$

is a Lie algebra

Generalizations

① $\mathfrak{g}$-bundle over Γ $\rightsquigarrow$ $\mathfrak{g}_\Sigma$, $\mathfrak{g}^*_\Sigma$ sections

$$c(\theta\theta\alpha, \phi\theta\beta) = \int_\Gamma \big((\alpha, d_{A_0} g) - (\beta, d_{A_0} f) \big)$$

d_{A_0} covariant derivative w.r.t. some connection

② If $(\mathfrak{g}, \langle,\rangle)$ quadratic

$$c_\Lambda(\theta\theta\alpha, \phi\theta\beta) := c(\theta\theta\alpha, \phi\theta\beta) + \Lambda \int \langle \alpha, \beta \rangle$$

$\rightsquigarrow$ $\hat{\mathfrak{g}}^\Lambda_\Gamma$

- $\hat{\mathfrak{g}}_\Gamma^\Lambda$ is related to 4D BF theory

  (with "cosmological" term for $\Lambda \neq 0$)

  It is worth being studied

- For 4D BF theory, one has to consider the Poisson manifold

$$\left(\hat{\mathfrak{g}}_\Gamma^\Lambda\right)^* = \underset{\substack{\cup\\ B}}{\Omega^2(\Sigma)\otimes\mathfrak{g}^*} \oplus \underset{\substack{\cup\\ A}}{\Omega^1(\Sigma)\otimes\mathfrak{g}}$$

and the Poisson submanifold

$$P = \{(A, B) : F_A + \Lambda B = 0\}$$

4D Gravity is related to the above:

- $\mathfrak{g} = so(3,1) \cong \Lambda^2 \mathbb{R}^4$ (or $so(4)$ for Euclidean gravity)

- A further constraints

$$Pf(B) = 0 \quad (\text{and } B \neq 0)$$

which defines a Poisson submanifold of P

Part II Poisson$_\infty$ structures from graded manifolds

A Poisson structure on a manifold M is the same a function S of degree 2 on $T^*[1]M$

s.t. $\{S, S\} = 0$

In fact, $C^\infty(T^*[1]M) = \Gamma(\wedge^\bullet TM)$

$$\{\ ,\ \} = [\ ,\ ]_{SN}$$

Generalization: allow M to be a graded manifold itself

$$\Rightarrow \quad S \rightsquigarrow \quad \pi = \pi_0 + \pi_1 + \pi_2 + \dots, \qquad \pi_k \in \Gamma(\Lambda^k TM)$$

$$\deg \pi_k = 2-k$$

$$\{f_1, \dots, f_k\}_k := \pi_k(df_1, \dots, df_k)$$

L_∞-structure by multiderivations $=: P_\infty$

<u>Notation</u> We call such (M, S) a BF^2V manifold

Further generalization:

1) Replace $T^*[1]M$ by any graded manifold $\mathcal{M}$ with symplectic form of degree 1.

2) Write $\mathcal{M}$ as $T^*[1]M$ (choice of polarization)

<u>Example</u> $\mathfrak{g}$ finite dimensional Lie algebra

$\Rightarrow$ $\mathfrak{g}^*$ Poisson manifold w/ KKS Poisson structure

$\rightsquigarrow$ $\mathcal{M} = T^*[1]\mathfrak{g}^* = \mathfrak{g}^* \oplus \mathfrak{g}[1]$

$\pi = \pi_2 = \pi_{KKS} \rightsquigarrow S$

We can also write $\mathcal{M} = T^*[1]\mathfrak{g}[1]$

$S \rightsquigarrow \pi = \pi_1 = \delta_{CE}$ Chevalley-Eilenberg

A further generalization:

- A graded Poisson algebra with $S \in A_2$, $\{S, S\} = 0$ (e.g. $C^\infty(T^*[1]M)$)

- splitting $A = h \oplus g$
  h, g Poisson subalgebras
  h abelian

  (e.g. $h = C^\infty(M)$
  $g = \Gamma(\Lambda^{>0} TM)$)

- Define derived brackets on h [T. Voronov]

$$\{f_1, \dots, f_k\}_k := P\{\dots\{\{S, f_1\}, f_2\}, \dots, f_k\}$$

$f_i \in h$ $\qquad P : A \to h$

<u>Claim</u> (BV) Field theories yield a BF^2V structure on their codimension-2 corners

Polarization $\rightsquigarrow$ Poisson_∞-structure

Pretty obvious in AKSZ theories

- Target $(Y, \omega_Y = d\alpha_Y, S_Y)$

symplectic of degree $n-1$ — degree n — $\{S, S\} = 0$

- Source $T^*[1]Z$

k-manifold

- bulk: $k = n \rightsquigarrow$ BV
- corner: $k = n-2 \rightsquigarrow$ BF^2V

- The two examples I gave at the beginning are of this type
    - Chern-Simons
    - BF_4
- Interesting non AKSZ example:

  4D Palatini-Cartan gravity

Thanks