The Fukaya category of the log symplectic sphere

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First Steps in Mirror Symmetry for Generalized Complex Geometry

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What is a Fukaya category?

- Invariant assigned to (nice) symplectic manifold
- Homological mirror symmetry (HMS):

$$\begin{array}{ccc} \text{symplectic mfd } (M,\omega) & & \text{complex mfd } (\hat{M}, I) \\ \Downarrow & & \Downarrow \\ \text{Fukaya category} & \underset{\substack{| \\ \text{equiv. of } A_\infty\text{-categories}}}{\cong} & \text{Derived cat. of coherent sheaves} \end{array}$$

<u>What is it (appr.)?</u>

Objects: (some) Lagrangian submfds

Morphisms: $\hom(L_0, L_1) \approx CF(L_0,L_1) := \bigoplus_{p \in L_0 \cap L_1} \underset{\substack{| \\ \text{field } (\mathbb{Z}_2)}}{\Lambda}\, p$ — transversality!

A_∞-operations: for ob. $L_0, \dots, L_k$, $k \geq 1$

$$m_k : CF(L_{k-1}, L_k) \times \dots \times CF(L_0, L_1) \longrightarrow CF(L_0, L_k)$$

subject to A_∞-relations.

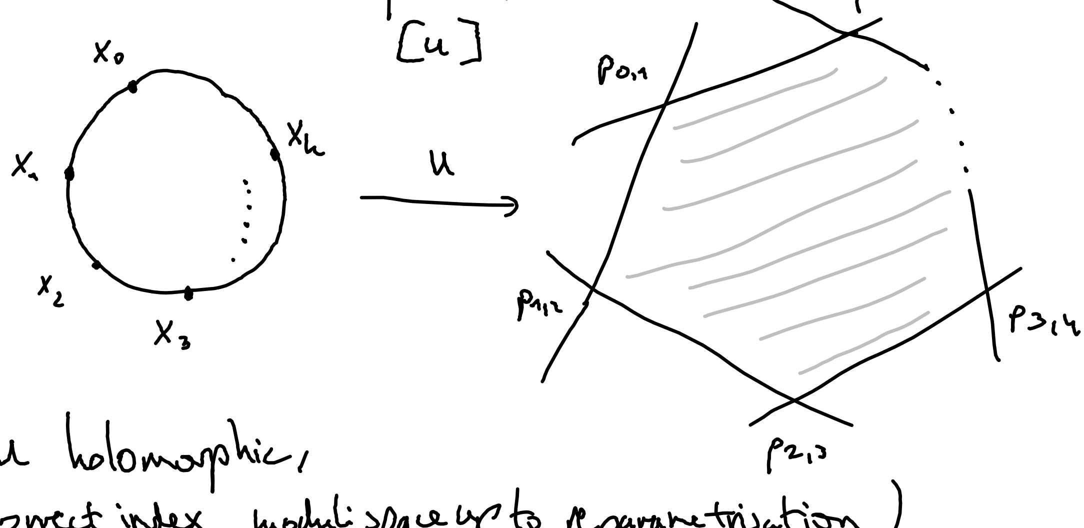
$k=1$: $m_1 = \partial$, $\partial^2 = 0$

$k=2$: composition of morphisms

$$m_2(a, m_2(b,c)) + m_2(m_2(a,b),c)$$
$$= \partial m_3(a,b,c) + m_3(\partial a, b, c) + m_3(a, \partial b, c) + m_3(a,b,\partial c)$$

<u>What are the m_k?</u>

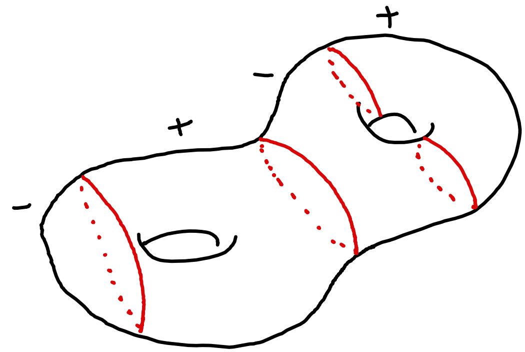
$$m_k(p_{k-1,k}, \dots, p_{0,1}) = \sum_{\substack{q \in L_0 \cap L_k \\ [u]}} \# \mathcal{M}(p_{0,1}, \dots, p_{k-1,k}, q; [u], J)\, q\, T^{\omega([u])}$$



(u holomorphic, correct index, moduli space up to reparametrisation)

Log symplectic surfaces

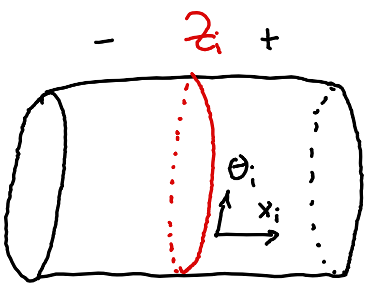
(b-symplectic, b-Poisson, topologically stable Poisson)



- (M^2, σ) oriented closed surface
- $\pi \in \Gamma(\Lambda^2 TM)$, $\pi \pitchfork \{0\}$

  $Z := \{\pi = 0\} = \bigsqcup_i Z_i$ circles

  $\omega := \pi^{-1}$ log symplectic form



$\pi = c\, x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial \theta_i}$, $\omega = \frac{1}{c} \frac{dx_i}{x_i} \wedge d\theta_i$

$d \log x_i$

- Hamiltonian v.f.: $X_f = \pi(df) = x_i \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial \theta_i} - \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial x_i} \right)$

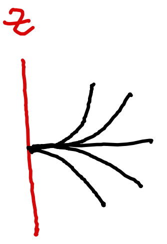
  $\Rightarrow$ Ham. Iso $|_Z = \mathrm{Id}_Z$

- Lagrangian submfds: Embedded circles $\pitchfork Z$,
  assume non-contractible with fixed Z

Constructing the Fukaya cat.

• For (M^2, σ, ω), fix vol. form ω_0 & $h \in C^\infty(M)$ s.t.
$\pi = h\,\omega_0^{-1}$,
h Morse with few critical points

• Make a Fukaya cat. for each $Z_n \subset Z$, $|Z_n| = n$,
$\mathrm{Fuk}_{Z_n}(M)$
Build collection of objects (=Lagrangians) $\mathcal{L}_{Z_n}$ s.t. $\forall \alpha \in \mathcal{L}_{Z_n}$:
– $\alpha \cap Z \subseteq Z_n$, $\alpha \cap \mathrm{crit}(h) = \emptyset$
– $h|_\alpha$ Morse
– near each $q_i \in Z_n$, all α agree
– $\hat{t} > 0$ small $\phi_h^{\hat t}(\alpha)$

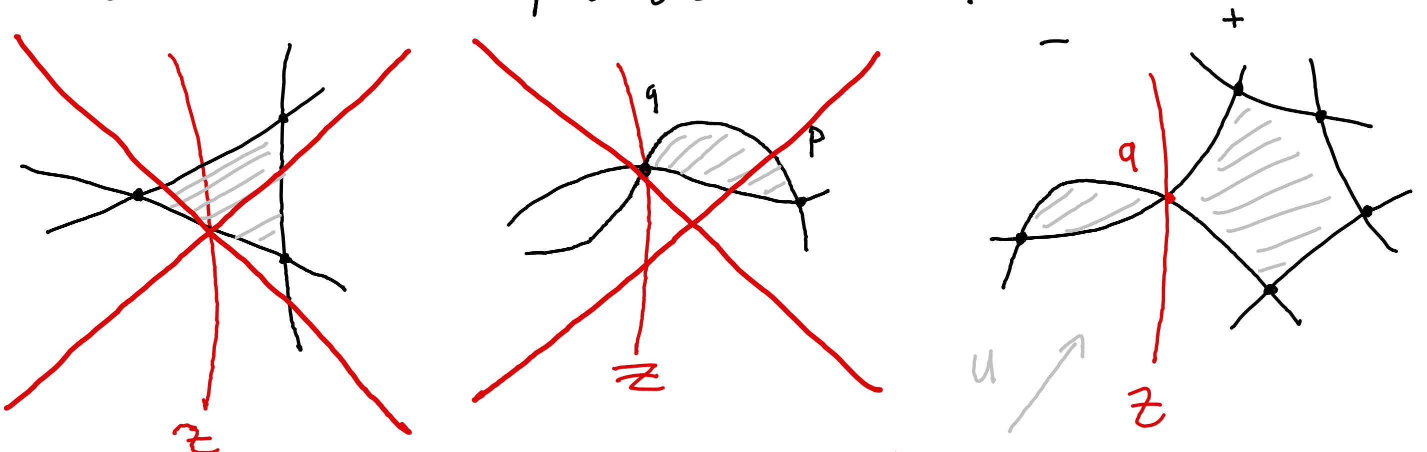


Wishlist / Almost-theorem

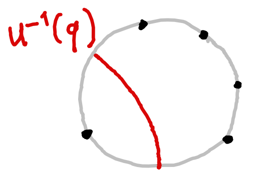
• $\mathrm{Fuk}_{Z_n}(M)$ is actually an A_∞-cat. (A_∞-rels. hold.)
• Different choices at each step. ⇝ A_∞-equiv. Fukaya cat.
($\mathrm{Fuk}_{Z_n}(M)$ is well-defined.)
• There is a finite set of (split) generators.

What is new compared to symplectic case?

• More Lagrangians / more generators
• Common intersection points in Z: $\alpha, \beta \in \mathcal{L}_{Z_n}$, $\alpha \cap \beta \cap Z \neq \emptyset$
• What about holomorphic discs near Z?



$q_i \in Z$ never interact! "inert"



"crossing disc"

Hamiltonian perturbation:

For $\alpha, \beta \in \mathcal{L}_{Z_n}$, $\hom(\alpha, \beta) := CF(\alpha, \phi_h^t(\beta))$, $t \ll \hat t$

$$m_k : CF(\phi_h^{t_{k-1}}(\alpha_{k-1}), \phi_h^{t_k}(\alpha_k)) \otimes \dots \otimes CF(\alpha_0, \phi_h^{t_1}(\alpha_1)) \longrightarrow CF(\alpha_0, \phi_h^{t_k}(\alpha_k))$$
$$0 < t_1 < \dots < t_k < \hat t$$

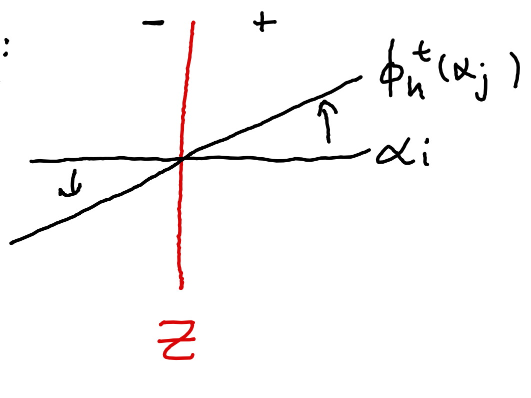
(no triple intersections!)

$\{\phi_h^t\}$ 1-parameter group of diffeos:

$$CF(\phi_h^{t_i}(\alpha_i), \phi_h^{t_j}(\alpha_j)) \cong CF(\alpha_i, \phi_h^{t_j - t_i}(\alpha_j)) = \hom(\alpha_i, \alpha_j)$$

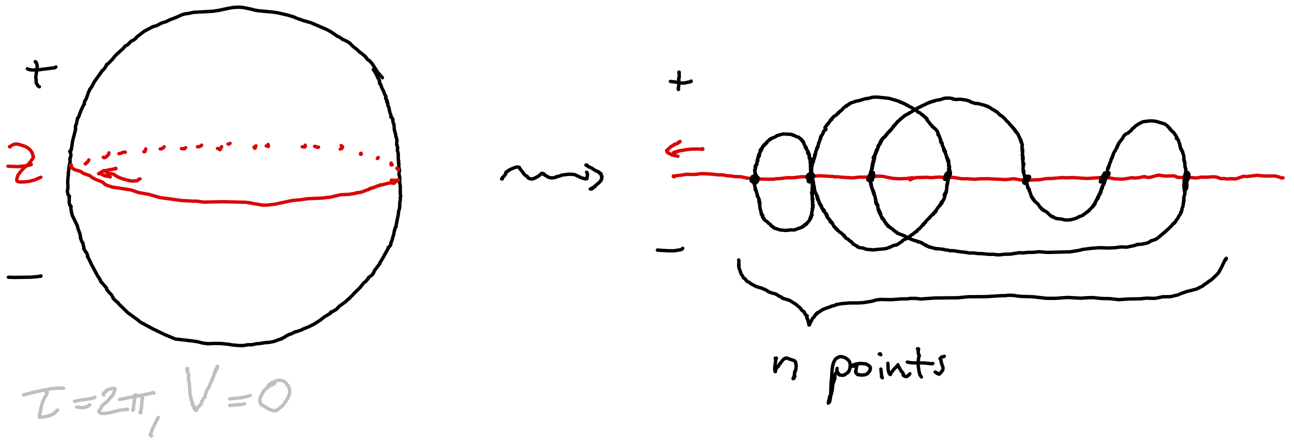
Ensures that A_∞-relations hold!

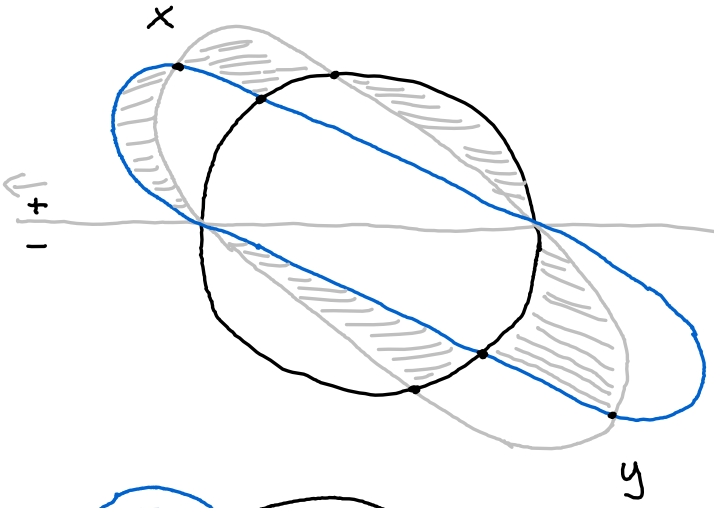
Near $q \in Z$:



$\Rightarrow$ In general $\hom(\alpha, \beta) \not\cong \hom(\beta, \alpha)$

Example: The standard log symplectic sphere



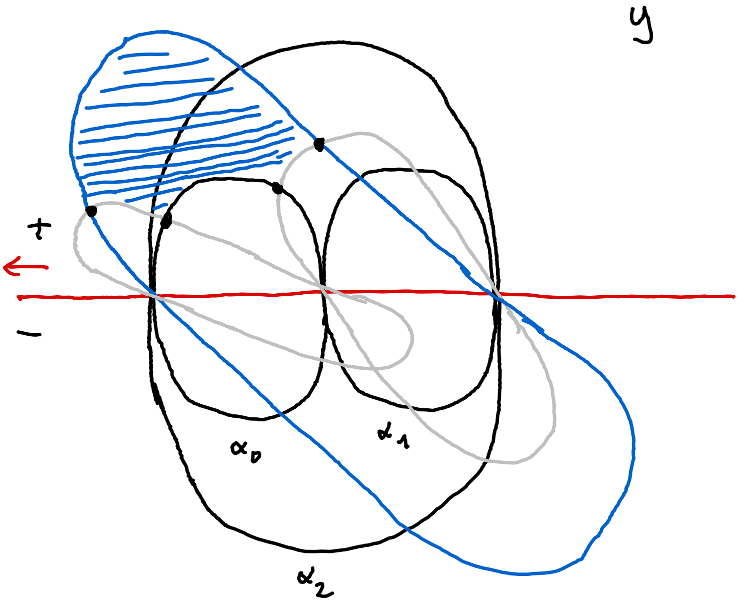


$x^2 = x$

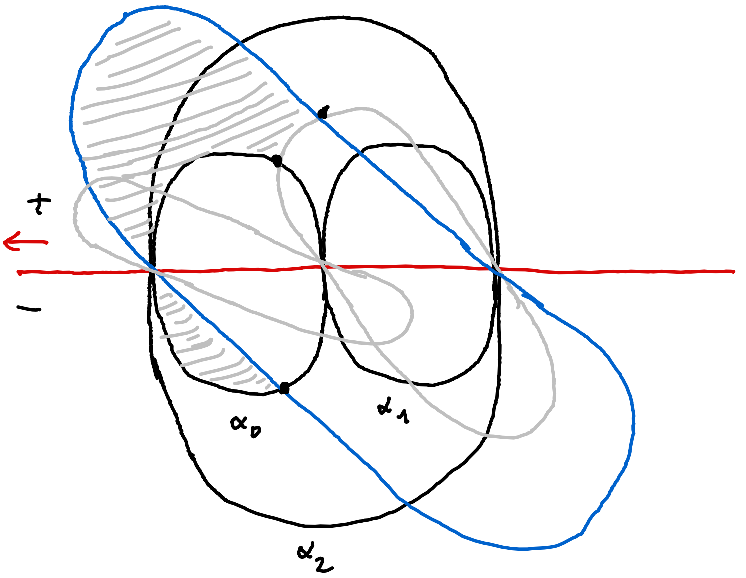
$xy = y$

$yx = y$

$y^2 = 0$



$m_3(x_{20}, x_{12}, x_{01}) = x_{00}$



$m_2(x_{12}, x_{01}) = y_{02}$

$m_2(y_{10}, x_{01}) = y_{00}$ . . .

$m_2(x_{21}, y_{10}) = y_{11}$

Outlook: Mirror symmetry?