Higher differential geometry and symplectic structures

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Amherst, Gone Fishing, March. 19 2023

Higher categorical structures

- Alan Weinstein 80's Birthday Conference in Paris Jun 19-23
- Dorothea Schlözer female postdoc program, deadline April 15th

- Source: theorectical physics needs to go to higher dimensions (3+1 dimension). For example, very roughly, higher dim. TQFT is a higher functor from the higher category of higher-cobordisms to that of higher-vector spaces.
- Lurie, Toen et al.: higher structures in algebraic geometry (around 00's) absorbs homotopy theory and category theory. It is very universal however also rather abstract.
- Baez et al., Poisson community, Stolz-Teichner program: higher structures in differential geometry. It involves more concrete models and more direct relation to math physics.

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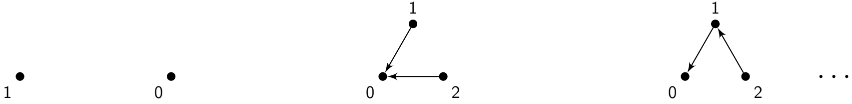
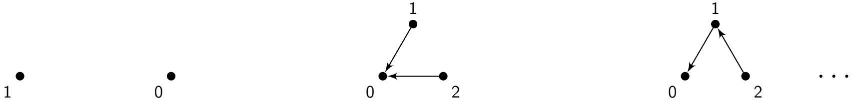


$$\Lambda^1_0(X) = X_0, \quad \Lambda^1_1(X) = X_0, \quad \Lambda^2_0(X) = X_1 \times_{d_1, X_0, d_1} X_1, \quad \Lambda^2_1(X) = X_1 \times_{d_0, X_0, d_1} X_1,$$

Horn



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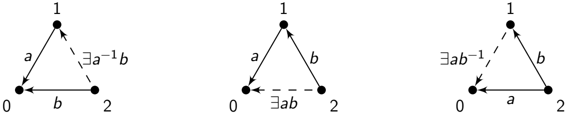

Kan(2, 0) Kan(2, 1) Kan(2, 2)

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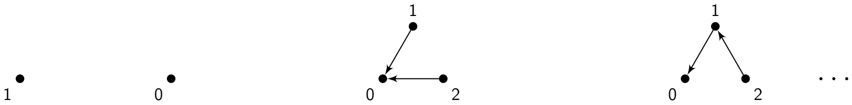

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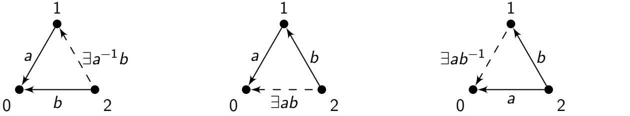

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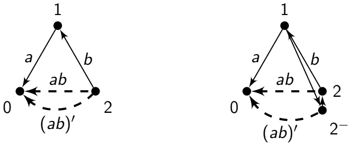

Kan(3, 1)

incomplete Category of fibrant objects (i.C.F.O.)

Let $\mathcal{C}$ be a category with finite products and terminal object $* \in \mathcal{C}$ equipped with two distinguished classes of morphisms called **weak equivalences** and **fibrations**. A morphism which is both a weak equivalence and a fibration is called an **acyclic fibration**. We say $\mathcal{C}$ is an **category of fibrant objects (CFO)** iff:

1. Every isomorphism in $\mathcal{C}$ is an acyclic fibration.
2. The class of weak equivalences satisfy "2 out of 3".
3. The composition of two fibrations is a fibration.
4. If the pullback along a fibration exists, then it is a fibration.
5. The pullback along an acyclic fibration exists, and is an acyclic fibration.
6. For any object $X \in \mathcal{C}$ there exists a (not necessarily functorial) **path object**.
7. All objects of $\mathcal{C}$ are **fibrant**. That is, for any $X \in \mathcal{C}$ the unique map $X \to *$ is a fibration.

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Let $\mathcal{C}$ be a category with finite products and terminal object $* \in \mathcal{C}$ equipped with two distinguished classes of morphisms called **weak equivalences** and **fibrations**. A morphism which is both a weak equivalence and a fibration is called an **acyclic fibration**. We say $\mathcal{C}$ is an **incomplete category of fibrant objects (iCFO)** iff:

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2. The class of weak equivalences satisfy "2 out of 3".
3. The composition of two fibrations is a fibration.
4. If the pullback along a fibration exists, then it is a fibration. That is, if $Y \xrightarrow{g} Z \xleftarrow{f} X$ is a diagram in $\mathcal{C}$ with f a fibration, and if $X \times_Z Y$ exists, then the induced projection $X \times_Z Y \to Y$ is a fibration.
5. The pullback along an acyclic fibration exists, and is an acyclic fibration. That is, if $Y \xrightarrow{g} Z \xleftarrow{f} X$ is a diagram in $\mathcal{C}$ with f an acyclic fibration, then the pullback $X \times_Z Y$ exists, and the induced projection $X \times_Z Y \to Y$ is an acyclic fibration.
6. For any object $X \in \mathcal{C}$ there exists a (not necessarily functorial) **path object**

Lemma (Brown's factorization Lemma for iCFO, Rogers-Zhu'20)

If $\mathcal{C}$ is an iCFO and $f : X \to Y$ is a morphism in $\mathcal{C}$, then f can be factored as $f = p \circ i$, where p is a fibration, and i is a weak equivalence which is a section (right inverse) of an acyclic fibration.

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Theorem (Rogers-Zhu'20)

The simplicial localization $L_W\mathcal{C}$ (or underlying ∞-category) of an iCFO $\mathcal{C}$ has a nice description in terms of the nerves of categories of spans. That is, for all objects $X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{L_W\mathcal{C}}(X, Y) \leftarrow N\mathrm{Span}_{w.e.}(X, Y) \leftarrow N\mathrm{Span}_{acyc.}(X, Y)$$

are weak equivalence of simplicial sets.

Integration v.s. Differentiation

Global $\xrightarrow{\textit{differentiation}}$ Infinitesimal

Global $\xleftarrow{\textit{integration}}$ Infinitesimal

Getzler'04	L_∞-group	$\longleftarrow$	nilpotent L_∞-algebra
Henriques'06	L_∞-group	$\longleftarrow$	L_∞-algebra
Tseng-Zhu'06	2-étale Lie 2-groupoid	$\xleftrightarrow{1-1}$	Lie algebroid
Ševera-Širaň'20	L_∞-groupoid	$\xleftarrow{\textit{local}}$	L_∞-algebroid$\|_U$
LFWRZ (now)	L_∞-groupoid	$\longrightarrow$	tangent complex

Differential Forms on $X_\bullet$

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\Omega^0(X_2) & \xrightarrow{d} & \Omega^1(X_2) & \xrightarrow{d} & \Omega^2(X_2) & \xrightarrow{d} & \dots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\Omega^0(X_1) & \xrightarrow{d} & \Omega^1(X_1) & \xrightarrow{d} & \Omega^2(X_1) & \xrightarrow{d} & \dots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\Omega^0(X_0) & \xrightarrow{d} & \Omega^1(X_0) & \xrightarrow{d} & \Omega^2(X_0) & \xrightarrow{d} & \dots
\end{array}
$$

Normalised Differential Forms on $X_\bullet$

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\hat\Omega^0(X_2) & \xrightarrow{d} & \hat\Omega^1(X_2) & \xrightarrow{d} & \hat\Omega^2(X_2) & \xrightarrow{d} & \dots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\hat\Omega^0(X_1) & \xrightarrow{d} & \hat\Omega^1(X_1) & \xrightarrow{d} & \hat\Omega^2(X_1) & \xrightarrow{d} & \dots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
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\end{array}
$$

$\omega \in \hat\Omega^k(X_l)$ iff $s_i^*\omega = 0$ for all $i = 0, \dots, k$.

For $x \in X_0$, $v \in \mathcal{T}_l(X_\bullet)_x$, $w \in \mathcal{T}_{m-l}(X_\bullet)_x$,

$$\lambda_x^{\omega_\bullet}(v, w) := \sum_{\sigma \in \mathrm{Sf}_{k,m-k}} (-1)^\sigma \omega_m(T(s_{\sigma(m-1)} \dots s_{\sigma(k)})v, T(s_{\sigma(k-1)} \dots s_{\sigma(0)})w)$$

where $\mathrm{Sf}_{k,m-k}$ is the set of $(k, m-k)$-shuffles, and $(-1)^\sigma$ is the permutation sign of σ.

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I.M. forms and shifted symplectic forms

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- $\lambda^{\omega_\bullet}$ is graded anti-symmetric;
- $\lambda^{\omega_\bullet}$ is infinitesimally multiplicative, that is,

$$\lambda^{\alpha_\bullet}(\partial u, w) + (-1)^{k+1}\lambda^{\alpha_\bullet}(u, \partial w) = 0.$$

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Definition

A m-shifted symplectic form on a Lie n-groupoid is a non-degenerate, closed m-shifted 2-form.

Examples for small m, n

m / n	0	1	2
0	Symp. Mfd.		
1	Symp. Stack	integrable Dirac Mfd.	BG
2		non-int Dirac Mfd.	Courant algebroid

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0	Symp. Mfd.		
1	Symp. Stack	Quasi Symp. Gpd.	BG
2		Quasi Symp. stacky Gpd.	Courant algebroid

Symp. Morita Equivalence

Definition (Cueca-Zhu'23)

Morita Equivalence of Lie n-groupoids $X_\bullet$ and $Y_\bullet$ is given by

$$X_\bullet \xleftarrow{f_\bullet} Z_\bullet \xrightarrow{g_\bullet} Y_\bullet$$

where $f_\bullet$ and $g_\bullet$ are acyclic fibrations (also called hypercovers defined in iCFO).

Symp. Morita Equivalence

Definition (Cueca-Zhu'23)

Symplectic Morita Equivalence of m-shifted symplectic Lie n-groupoids $(X_\bullet, \alpha_\bullet)$ and $(Y_\bullet, \beta_\bullet)$ is given by

$$(X_\bullet, \alpha_\bullet) \xleftarrow{f_\bullet} (Z_\bullet, \phi_\bullet) \xrightarrow{g_\bullet} (Y_\bullet, \beta_\bullet)$$

where $f_\bullet$ and $g_\bullet$ are acyclic fibrations (also called hypercovers defined in iCFO). Moreover, $\phi_\bullet$ is a $m-1$-shifted form and $f_\bullet^*\alpha_\bullet - g_\bullet^*\beta_\bullet = D\phi_\bullet$.

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Theorem (Cueca-Zhu'23)

When $m = n = 1$, this symplectic Morita Equivalence coincides with the one defined in Xu'04 via Hamiltonian bimodules.

BG as 2-shifted symplectic stack

Theorem (Cueca-Zhu'23)

The 2-shifted sympolectic stack BG has the following symplectic Morita Equivalent models

$$
\begin{array}{ccccc}
\bullet & \Longleftarrow & (G, 1/2\Theta) & \Longleftarrow & (G \times G, 1/2\Omega) \dots \\
 & & \uparrow{\scriptstyle ev_1} & & \uparrow{\scriptstyle ev_2} \\
\bullet & \Longleftarrow & (PG, \omega) & \Longleftarrow & \Omega G \dots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\bullet & \Longleftarrow & PG & \Longleftarrow & (\Omega G, \omega) \dots
\end{array}
$$

where $\Theta \in \Omega^3(G)$ is the Cartan 3-form, $\Omega \in \Omega^2(G \times G)$ is the Brylinski-Weinstein 2-form, and ω is the Segal's 2-form.

Acyclic Fibration, Weak Equivalence

Definition (Zhu'09)

A simplicial morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a **acyclic fibration or hypercover** if the maps

$$((d_0, \cdots, d_i), f_i) : X_i \to \partial_i(X_\bullet) \times_{\partial_i(Y_\bullet)} Y_i = \mathrm{Hom}(\partial\Delta[i] \to \Delta[i], X_\bullet \to Y_\bullet)$$

are surjective submersions for $0 \leq i < n$ and an isomorphism for $i = n$.

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Definition (Behrend-Getzler'18, Rogers-Zhu'20)

A simplicial morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a **weak equivalence** if

$$r_i : \mathrm{Hom}(\Delta[i] \to \Delta[i+1], X_\bullet \to Y_\bullet) \to \mathrm{Hom}(\partial\Delta[i] \to \Lambda[i+1, i+1], X_\bullet \to Y_\bullet)$$

is a surjective submersion for $i < n$ and isomorphism for $i \ge n$.

Integration and Differentiation

Integration:

- L_∞-algebras by Henriques'08,
- nilpotent L_∞-algebras by Getzler'09,
- L_∞-algebroids (local result) by Ševera-Širaň'20

Differentiation:

- A formalization of the notion Lie differentiation in higher geometry has been given by Lurie (deformation context), worked out in diff. Geom. setting by Joost Nuiten, but the result is abstract.
- Ševera sketches an explicit differentiation functor with an idea inspired by Konsevich 2006. But the result is only a presheaf.
- Differentiation to tangent complex [Fernandes-Li-Ryvkin-Wessel-Zhu].