Weightings for Lie groupoids and Lie algebroids

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Work in progress with Eckhard Meinrenken

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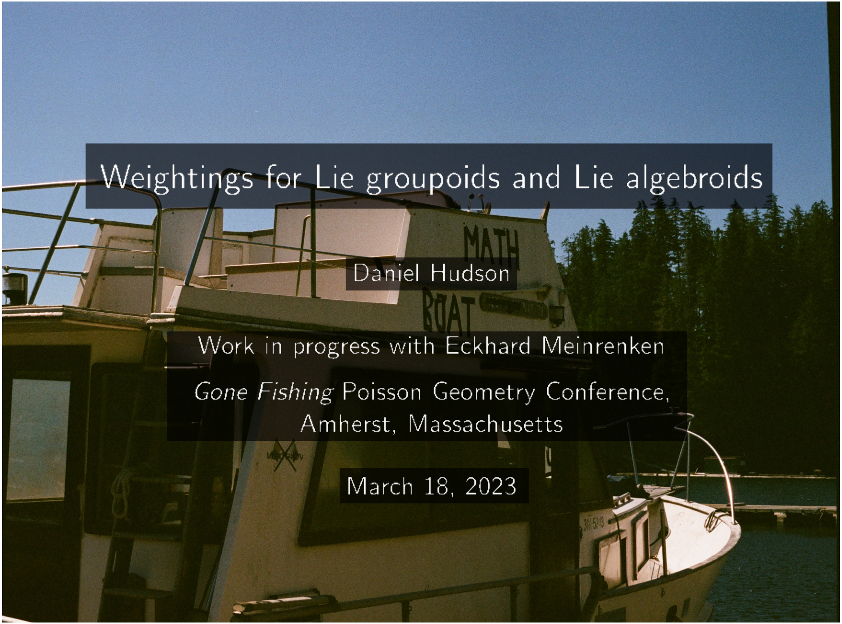

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2. multiplicative weightings for Lie groupoids
3. infinitesimally multiplicative weightings for Lie algebroids
4. differentiation of multiplicative weightings

Weightings

Definition (Loizides-Meinrenken, 2020)

Fix weights $0 \leq w_1 \leq \cdots \leq w_n \leq r \in \mathbb{Z}$.

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Fix weights $0 \leq w_1 \leq \cdots \leq w_n \leq r \in \mathbb{Z}$. A filtration

$$C^\infty(M) = C^\infty(M)_{(0)} \supseteq C^\infty(M)_{(1)} \supseteq C^\infty(M)_{(2)} \supseteq \cdots \tag{1}$$

is a *weighting* of M if locally one has coordinates x_a so that

$$f \in C^\infty(M)_{(i)} \iff f(x) = \sum_{|\alpha| \geq i} x^\alpha f_\alpha(x),$$

where $|\alpha| = \sum_{i=1}^n \alpha_i w_i$.

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$$f^* C^\infty(M')_{(i)} \subseteq C^\infty(M)_{(i)} \quad \forall i.$$

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Remark: $C^\infty(M)_{(1)} = \mathcal{I}_N$ for a closed submanifold $N \subseteq M$. (M, N) is a *weighted manifold pair*.

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The trivial weighting of M along N is given by

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$(M, N) = (\mathbb{R}^2, \{0\})$, $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$:

$$C^\infty(\mathbb{R}^3) \supseteq \langle x, y \rangle \supseteq \langle x^2, y \rangle \supseteq \langle x^3, y \rangle \supseteq \langle x^4, xy, y^2 \rangle \supseteq \cdots$$

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2. filtration of k-forms:

$$\Omega^k(M) = \Omega^k(M)_{(0)} \supseteq \Omega^k(M)_{(1)} \supseteq$$

Weighted Normal Bundle

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$$\mathrm{gr}(C^\infty(M)) = \bigoplus_{i \geq 0} C^\infty(M)_{(i)}/C^\infty(M)_{(i+1)}.$$

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Theorem (Loizides-Meinrenken, 2020, c.f. Haj-Higson, 2018)

The weighted normal bundle

$$\nu_{\mathcal{W}}(M, N) := \mathrm{Hom}_{alg}(\mathrm{gr}(C^\infty(M)), \mathbb{R})$$

is a graded bundle over N of dimension $\dim(M)$ *s.t.*

$$\mathrm{gr}(C^\infty(M)) \subseteq C^\infty(\nu_{\mathcal{W}}(M, N)).$$

This construction is functorial for weighted morphisms.

Weighted Deformation Space

For a weighted pair (M, N) let

$$\mathrm{Rees}(C^\infty(M)) = \left\{ \sum_{i\in\mathbb{Z}} f_i z^{-i} : f_i \in C^\infty(M)_{(i)} \right\} \subseteq C^\infty(M)[z, z^{-1}].$$

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Theorem (Loizides-Meinrenken, 2020, c.f. Haj-Higson, 2018)

The *weighted deformation space*

$$\delta_{\mathcal{W}}(M, N) := \mathrm{Hom}_{alg}(\mathrm{Rees}(C^\infty(M)), \mathbb{R})$$

has C^∞ *structure of dimension* $\dim(M) + 1$ *s.t*

$$\mathrm{Rees}(C^\infty(M)) \subseteq C^\infty(\delta_{\mathcal{W}}(M, N)).$$

This construction is functorial for weighted morphisms.

Example

There is a surjective submersion $\pi : \delta_{\mathcal{W}}(M, N) \to \mathbb{R}$ giving the decomposition

$$\delta_{\mathcal{W}}(M, N) = \nu_{\mathcal{W}}(M, N) \bigsqcup M \times \mathbb{R}^{\times}.$$

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Example

1. $\delta(M \times M, \Delta_M) = TM \sqcup (M \times M \times \mathbb{R}^{\times})$, Connes' tangent groupoid.

Question

Suppose that $G \rightrightarrows M$ is weighted along H. What are the correct conditions on the weighting so that $\nu_{\mathcal{W}}(G, H)$ and $\delta_{\mathcal{W}}(G, H)$ are Lie groupoids?

Multiplicative Weightings

Definition

Suppose that $G \rightrightarrows M$ is weighted along $H \rightrightarrows N$. (G, H) is *multiplicatively weighted* if

1. $M \subseteq G$ is a weighted submanifold,
2. $s, t : G \to M$ are weighted submersions,
3. $\mathrm{mult} : G^{(2)} \to G$ is a weighted morphism,
4. $\mathrm{inv} : G \to G$ is a weighted morphism.

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R inherits a weighting along $R \cap N$ via the filtration

$$C^\infty(R)_{(i)} := C^\infty(M)_{(i)}/(\mathcal{I}_R \cap C^\infty(M)_{(i)})$$

s.t. $R \hookrightarrow M$ is weighted.

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Weighted Submanifolds

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- M is trivially weighted along N, then $R \subseteq (M, N)$ is a weighted submanifold iff R intersects N cleanly.
- $\mathbb{R}^2$, $\mathrm{wt}(x) = 1$ and $\mathrm{wt}(y) = 3$. The curve $y = x^2$ is *not* a weighted submanifold of $(\mathbb{R}^2, \{0\})$.

Weighted Morphisms

Definitions

A weighted morphism $f : (M, N) \to (M', N')$ is a *weighted submersion* if the induced map

$$\delta_{\mathcal{W}}(f) : \delta_{\mathcal{W}}(M, N) \to \delta_{\mathcal{W}}(M', N')$$

is a submersion.

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Proposition

If $f : (M, N) \to (M', N')$ is a weighted submersion and R' is a weighted submanifold of (M', N'), then $f^{-1}(R')$ is a weighted submanifold of (M, N).

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Multiplicative Weightings

Theorem

If $G \rightrightarrows M$ is multiplicatively weighted along $H \rightrightarrows N$, then

$$\nu_{\mathcal{W}}(G, H) \rightrightarrows \nu_{\mathcal{W}}(M, N) \quad \textit{and} \quad \delta_{\mathcal{W}}(G, H) \rightrightarrows \delta_{\mathcal{W}}(M, N)$$

are Lie groupoids.

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- Osculating tangent groupoid:

$$\delta_{\mathcal{W}}(\mathrm{Pair}(M), \Delta_M) = \mathbb{T}_H M.$$

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What is the infinitesimal analogue of multiplicative weightings?

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$$\cdots \supseteq \Gamma(V)_{(i)} \supseteq \Gamma(V)_{(i+1)} \supseteq \cdots$$

s.t. locally one has a frame σ_a s.t.

$$\sigma \in \Gamma(V)_{(i)} \iff \sigma = \sum_{a=1}^{k} f_a \sigma_a$$

with $f_a \in C^\infty(M)_{(i-v_a)}$.

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Dual weighting & Weighted morphisms

If V is linearly weighted then V^* is linearly weighted with

$$\Gamma(V^*)_{(i)} = \{\tau \in \Gamma(V^*) : \forall j,\ \sigma \in \Gamma(V)_{(j)} \implies \langle \tau, \sigma \rangle \in C^\infty(M)_{(i+j)}\}.$$

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A $\mathcal{VB}$-morphism $\varphi : V \to W$ is *weighted* if the base map $\varphi|_M : M \to M'$ is weighted and

$$\varphi^* : \Gamma(W^*)_{(i)} \to \Gamma(V^*)_{(i)}.$$

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The space

$$\nu_{\mathcal{W}}(V) := \{(\varphi,\psi) : \varphi \in \nu_{\mathcal{W}}(M,N), \psi \in \mathrm{Hom}_{\varphi}(\mathrm{gr}(\Gamma(V^*)), \mathbb{R})\}$$

is a smooth vector bundle over $\nu_{\mathcal{W}}(M,N) = \mathrm{Hom}_{alg}(\mathrm{gr}(C^\infty(M)), \mathbb{R})$. This construction is functorial for weighted $\mathcal{VB}$-morphisms

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The weighted deformation bundle

$$\delta_{\mathcal{W}}(V) := \{(\varphi, \psi) : \varphi \in \nu_{\mathcal{W}}(M, N), \psi \in \mathrm{Hom}_{\varphi}(\mathrm{Rees}(\Gamma(V^*)), \mathbb{R})\}$$

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3. the Poisson bivector on A^* has filtration degree zero.

We say A is *infinitesimally multiplicatively weighted* in this case. A *weighted $\mathcal{LA}$-morphism* is weighted $\mathcal{VB}$-morphism $\varphi : A \to B$ s.t.

$$\varphi^* : \Gamma(\wedge^{\bullet} B^*) \to \Gamma(\wedge^{\bullet} A^*)$$

is a chain map.

Infinitesimally Multiplicative Weightings

Theorem

If $A \Rightarrow M$ is infinitesimally multiplicatively weighted then

$$\nu_{\mathcal{W}}(A) \Rightarrow \nu_{\mathcal{W}}(M, N) \quad \textit{and} \quad \delta_{\mathcal{W}}(A) \Rightarrow \delta_{\mathcal{W}}(M, N)$$

are Lie algebroids. This construction is functorial for weighted $\mathcal{LA}$-morphisms.

Differentiation of Multiplicative Weightings

Theorem

If $G \rightrightarrows M$ is multiplicatively weighted then $\mathrm{Lie}(G) \Rightarrow M$ obtains a canonical infinitesimally multiplicative weighting. One has that

$$\mathrm{Lie}(\nu_{\mathcal{W}}(G,H)) = \nu_{\mathcal{W}}(\mathrm{Lie}(G)) \quad \textit{and} \quad \mathrm{Lie}(\delta_{\mathcal{W}}(G,H)) = \delta_{\mathcal{W}}(\mathrm{Lie}(G))$$

Thanks for your attention!