On super cluster algebras based on super Plücker and super Ptolemy relations

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Introduction

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Introduction

The work is motivated by the problem of defining super cluster algebras, which has attracted a lot of attention in recent years.

The current state of the art is the definitions of Ovsienko [3] that were later modified into Ovsienko-Shapiro's definition [4], as well as Li-Mixco-Ransingh-Srivastava's definition [1]. Ovsienko and Shapiro note the problem of the absence of any mutation of odd variables in their definition. The Li-Mixco-Ransingh-Srivastava's definition requires testing against expected properties and examples.

Such examples come by generalizing to the super case some of the classical model examples. Known results in this direction come from supergeometry. Such are Musiker-Ovenhouse-Zhang's [2] super cluster algebra structure for the global coordinates on the decorated super Teichmüller space (based on the super Ptolemy relation of Penner-Zeitlin [6]), and Shemyakova-Voronov's [8] cluster algebra construction based on their super Plücker relation for the homogeneous coordinates of the super Grassmannian $Gr_{2|0}(n|1)$. The results of these two works have similarities and need to be compared.

Our results in this field.

1. Defined super analog of the classical Plücker embedding for the Graßmann supermanifold $Gr_{r|s}(m|n)$. The target space turned out to be the weighted projective space:

$$P_{1,-1}\left(\Lambda^{r|s}(V) \oplus \Lambda^{s|r}(\Pi V)\right)$$

2. Super Plücker coordinates and relations for $Gr_{r|s}(n|m)$.
3. Super cluster structures for $Gr_{2|0}(\mathbb{R}^{n|1})$. The exchange graph structure is understood.
4. Better Plücker coordinates and simpler relations for the dual case $Gr_{n-r|1}(n|1)$.
5. A possibly important observation about the super Ptolemy relations of Penner-Zeitlin [6] used by Musiker-Ovenhouse-Zhang [2] for their super cluster structure. By a change of variables that we propose here, the super Ptolemy relation transforms into the classical Ptolemy relation.

Classical Plücker embedding and coordinates

A real Grassmann manifold $Gr_r(V) = Gr_r(n)$ is the space of all r-dimensional subspaces of $V = \mathbb{R}^n$. Let $L \in Gr_r(n)$ and $\{u_i\}$ is its basis. Then $u_1 \wedge \cdots \wedge u_r \neq 0$ and does not depend on the choice of basis, up to multiplication by a non-zero scalar. Therefore, we have a well-defined map

$$\mathfrak{pl\ddot{u}ck} : Gr_r(V) \to P(\Lambda^r V)\,, \; L \mapsto u_1 \wedge \cdots \wedge u_r\,.$$

$$\begin{aligned}
u_1 \wedge \cdots \wedge u_r &= u_1^{a_1} e_{a_1} \wedge \cdots \wedge u_r^{a_r} e_{a_r} \\
&= \sum_{1 \le a_1 < \cdots < a_r \le n} \begin{vmatrix} u_1^{a_1} & \dots & u_1^{a_r} \\ \vdots & & \vdots \\ u_r^{a_1} & \dots & u_r^{a_r} \end{vmatrix} e_{a_1} \wedge \cdots \wedge e_{a_r} \\
&= \sum_{1 \le a_1 < \cdots < a_r \le n} P^{a_1 \dots a_r} e_{a_1} \wedge \cdots \wedge e_{a_r}\,.
\end{aligned}$$

We get the total of $\binom{n}{r} = \dim \Lambda^r V$ of minors $P^{a_1 \dots a_r}$ that are called Plücker coordinates.

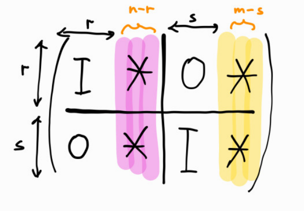
One can show that it is an embedding and the image is described by *Plücker relations*

$$P^{\underline{a}(a_r \leftarrow b)} P^{\underline{c}} = \sum_{j=1}^{r} P^{\underline{a}(a_r \leftarrow c_j)} P^{\underline{c}(c_j \leftarrow b)} .$$

Here $\underline{a} = a_1 \ldots a_r$, $\underline{a}(a_r \leftarrow b)$ stands for b replacing a_r, and so on. (Thus this construction realize $Gr_r(n)$ as a projective variety.)
It is convenient to extend $P^{a_1 \cdots a_r}$ by anti-symmetry to the case of arbitrary ordered indices.

Super Plücker embedding

Super Graßmannian. V is a super space of dimension $n|m$.
$Gr_{r|s}(V)$ is the space of $r|s$-dimensional subspaces of V with odd coefficients allowed.
Every $L \in Gr_{r|s}(V)$ is spanned by r even and s odd vectors.
Let U be the corresponding even $r|s \times n|m$ matrix.
By the left action of $GL(r|s)$, some columns of U can be "normalized". E.g.



So, $\dim Gr_{r|s}(n|m) = (r|s) \cdot (n|m - r|s) = r(n-r) + s(m-s) | r(m-s) + s(n-r)$.

The target space.

Algebraic construction? I.e. the quotient $\Lambda(V) = \oplus_{k=0}^{\infty}\Lambda^k(V)$ of the tensor algebra $T(V)$ by anticommutativity conditions: $\boldsymbol{e}_i \wedge \boldsymbol{e}_j = -\boldsymbol{e}_j \wedge \boldsymbol{e}_i$, $\boldsymbol{e}_{\hat\alpha} \wedge \boldsymbol{e}_i = -\boldsymbol{e}_i \wedge \boldsymbol{e}_{\hat\alpha}$, $\boldsymbol{e}_{\hat\alpha} \wedge \boldsymbol{e}_{\hat\beta} = \boldsymbol{e}_{\hat\beta} \wedge \boldsymbol{e}_{\hat\alpha}$?

Let e.g. $L = \mathrm{span}(u_1, \ldots, u_r, u_{\hat 1}, \ldots, u_{\hat s})$, what if we try map

$$L \mapsto u_1 \wedge \cdots \wedge u_r \wedge u_{\hat 1} \wedge \cdots \wedge u_{\hat s} \in \Lambda^{r+s}(V)\ ?$$

Does not work if $s \neq 0$. For example, let u_1 be an even and $u_{\hat 1}$ and $u_{\hat 2}$ be odd vectors. Then if we change basis,

$$u_1 \wedge (3u_{\hat 1} + 2u_{\hat 2}) \wedge u_{\hat 2} = 3u_1 \wedge u_{\hat 1} \wedge u_{\hat 2} + 2u_1 \wedge \underbrace{u_{\hat 2} \wedge u_{\hat 2}}_{\text{not zero}} \tag{1}$$

Indeed, we recall that for super vector space there is no top power among exterior powers.
This construction will work for the case of $Gr_{r|0}(n|m)$.

Voronov-Zorich construction of $\Lambda^{r|s}(V)$.

$\Lambda^{r|s}(V)$ consists of functions $F = F(p^1, \ldots, p^r | p^{\hat{1}}, \ldots, p^{\hat{s}})$ of r even and s odd covectors s.t.

1. $F(p \cdot h) = F(p) \cdot \mathrm{Ber}(h), \quad h \in GL(r|s)$
2. $\frac{\partial^2 F}{\partial p_a^i \partial p_b^j} + (-1)^{\tilde{i}\tilde{j}+\tilde{a}(\tilde{i}+\tilde{j})} \frac{\partial^2 F}{\partial p_a^j \partial p_b^i} = 0$

Notes:

- $\Lambda^{r|0}(V)$ can be identified with $\Lambda^r(V)$.
- (1) implies homogeneity of degree 1 in even and of degree -1 in odd covectors; anti-symmetry within each group.
- (2) implies multi linearity within even arguments.
- "Super wedge product": $F = [u_1, \ldots, u_r | u_{\hat{1}}, \ldots, u_{\hat{s}}]$ is defined by $F(p^1, \ldots, p^r | p^{\hat{1}}, \ldots, p^{\hat{s}}) := \mathrm{Ber}\langle u_i, p^j \rangle$ (Schwarz/Khudaverdian/Belopolsky).

Super Plücker embedding: algebraic case. The case of $r|0$ planes in $n|m$ space.

Theorem [SV'2021]. 1. Algebraic approach gives $\mathfrak{pl\ddot{u}ck}$ which is indeed an embedding.

$$\mathfrak{pl\ddot{u}ck} : \mathrm{span}\{\underbrace{u_1, \ldots, u_r}_{\text{all even}}\} \mapsto [u_1 \wedge \cdots \wedge u_r] \in P(\Lambda^r(V)).$$

2. Even $0 \neq T \in \Lambda^r(V)$ is simple $\iff$ its components satisfy "super Plücker relations":

$$(-1)^{\tilde{b}(\tilde{a}_1+\ldots+\tilde{a}_{r-1}+\tilde{c}_1+\ldots+\tilde{c}_r)} T^{\underline{a}(a_r \leftarrow b)} T^{\underline{c}}$$
$$= \sum_{j=1}^{r} T^{\underline{a}(a_r \leftarrow c_j)} T^{\underline{c}(c_j \leftarrow b)} (-1)^{\tilde{b}(\tilde{c}_1+\ldots+\tilde{c}_{j-1})+\tilde{c}_j(\tilde{a}_1+\ldots+\tilde{a}_{r-1}+\tilde{c}_{j+1}+\ldots+\tilde{c}_r)}$$

Here $\underline{a} = a_1 \ldots a_r$, $\underline{a}(a_i \leftarrow b)$ stands for b replacing a_i, and so on.

Relations for $Gr_{2|0}(4|1)$ were also found by Cervantez-Fiorezi-Lledó, 2011.

Super Plücker embedding: general case. $r|s$ planes in $n|m$ space.

First (not final) definition of super Plücker map:

$$\mathfrak{plück} : \ L \mapsto \mathfrak{plück}_L = \underbrace{[u_1, \ldots, u_r | u_{\hat{1}}, \ldots, u_{\hat{s}}]}_{\text{super analog of wedge product}}$$

(up to a multiplicative constant). So, indeed a well-defined map. In matrix form:

$$\mathfrak{plück}_L(P) = \operatorname{Ber}(UP) \,.$$

However, it does not yet give an embedding!

Example. Can we reconstruct an $L \in Gr_{1|1}(2|2)$ by $\mathfrak{pl\ddot{u}ck}_L$?

Fix a basis in V: $\underbrace{e_1, e_2}_{\text{even}}, \underbrace{e_{\hat{1}}, e_{\hat{2}}}_{\text{odd}}$. Let $L = \mathrm{span}(u_1 | u_{\hat{1}})$,

$$
\begin{aligned}
u_1 &= x e_1 + z e_2 + \xi e_{\hat{1}} + \nu e_{\hat{2}} \quad \text{even} \\
u_{\hat{1}} &= \eta e_1 + \alpha e_2 + y e_{\hat{1}} + t e_{\hat{2}} \quad \text{odd}
\end{aligned}
$$

where “Greek” are odd, “Latin” are even, and “hatted” indices means odd. It is always possible to normalize by choosing u_1 and $u_{\hat{1}}$, say

$$
\begin{aligned}
u_1 &= x e_1 + 1 \cdot e_2 + \xi e_{\hat{1}} + 0 \cdot e_{\hat{2}} \\
u_{\hat{1}} &= \eta e_1 + 0 \cdot e_2 + y e_{\hat{1}} + 1 \cdot e_{\hat{2}}
\end{aligned}
$$

$$F(p^1|p^{\hat{1}}) = \mathfrak{plück}_L(p^1|p^{\hat{1}}) = \underbrace{[u_1|u_{\hat{1}}]}_{F}(\underbrace{p^1}_{\text{even}} \,|\, \underbrace{p^{\hat{1}}}_{\text{odd}}) =$$

$$\operatorname{Ber}\left(\left(\begin{array}{cc|cc} x & 1 & \xi & 0 \\ \hline \eta & 0 & y & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} p_1 & \pi_1 \\ p_2 & \pi_2 \\ \hline p_{\hat{1}} & \pi_{\hat{1}} \\ p_{\hat{2}} & \pi_{\hat{2}} \end{array}\right)\right) =$$

$$\frac{xp_1 + p_2 + \xi p_{\hat{1}} - (x\pi_1 + \pi_2 + \xi\pi_{\hat{1}})(\eta\pi_1 + y\pi_{\hat{1}} + \pi_{\hat{2}})^{-1}(\eta p_1 + yp_{\hat{1}} + p_{\hat{2}})}{\eta\pi_1 + y\pi_{\hat{1}} + \pi_{\hat{2}}}$$

Now we use this formula for $F = \mathfrak{plück}_L = [u_1|u_{\hat{1}}]$ to evaluate it at different suitable covectors.

Our goal is to restore x, ξ, η, and y.

$$x = F(e^1|e^{\hat{2}}) = \operatorname{Ber}\left(\begin{array}{c|c} x & 0 \\ \hline \eta & 1 \end{array}\right) =: T^{1|\hat{2}}$$

$$\frac{1}{y} = F(e^2|e^{\hat{1}}) = \operatorname{Ber}\left(\begin{array}{c|c} 1 & \xi \\ \hline 0 & y \end{array}\right) =: T^{2|\hat{1}}$$

The only way to restore ξ is to evaluate $F(\underbrace{e^{\hat{1}}}_{\text{odd}}|e^{\hat{2}})$. We extend F by linearity to the tuple of covectors of this form.

Then $\xi = F(e^{\hat{1}}|e^{\hat{2}})$ $= \operatorname{Ber}\left(\begin{array}{c|c} \xi & 0 \\ \hline y & 1 \end{array}\right) =: T^{\hat{1}|\hat{2}}$.

To restore η we need make the odd coefficient at η in the numerator zero (because it is odd) and so to evaluate we need $F(e^2|\underbrace{e^1}_{\text{even}})$. However, no linearity in odd slot. The solution is to use the inverse Berezinian:

$\eta = F^*(e^2|e^1)$ $= \operatorname{Ber}^*\left(\begin{array}{c|c} 1 & x \\ \hline 0 & \eta \end{array}\right) =: T^{2|1}$.

(Notion $\operatorname{Ber}^*(A) = \operatorname{Ber}(A^\Pi)$ is due to Bergvelt-Rabin.) (One can read about Berezinians for such “wrong” matrices in [7])

<u>Super Plücker final definition [SV'2021].</u> For $L \in Gr_{r|s}(n|m)$,
$\mathrm{Plück}(L) = [\mathfrak{plück}_L, \mathfrak{plück}^*_L] \in P_{1,-1}\left(\Lambda^{r|s}(V) \oplus \Lambda^{s|r}(\Pi V)\right)$.

<u>Theorem [SV'2021].</u> Embedding. Super Plücker relations:

$$T^{\underline{a}|\underline{\hat{\mu}}}\, T^{*\underline{a}|\underline{\hat{\mu}}} = 1$$

$$\mathrm{Ber}\left(\begin{array}{c|c} T^{\underline{a}(a_i \leftarrow b_j)|\underline{\hat{\mu}}} & T^{\underline{a}(a_i \leftarrow \hat{\nu}_\beta)|\underline{\hat{\mu}}} \\ \hline T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow b_j)} & T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow \hat{\nu}_\beta)} \end{array}\right)_{\substack{i,j=1\dots r,\\ \alpha,\beta=1\dots s}} = \left(T^{\underline{a}|\underline{\hat{\mu}}}\right)^{r+s-1} T^{\underline{b}|\underline{\hat{\nu}}}$$

$$\mathrm{Ber}\left(\begin{array}{c:c|c} T^{\underline{a}(a_i \leftarrow b_j)|\underline{\hat{\mu}}} & T^{\underline{a}(a_i \leftarrow \hat{\lambda})|\underline{\hat{\mu}}} & T^{\underline{a}(a_i \leftarrow \hat{\nu}_\beta)|\underline{\hat{\mu}}} \\ \hline T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow b_j)} & T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow \hat{\lambda})} & T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow \hat{\nu}_\beta)} \end{array}\right)_{\substack{i=1\dots r,\\ j=1\dots r-1,\\ \alpha,\beta=1\dots s}}$$

$$= \left(T^{\underline{a}|\underline{\hat{\mu}}}\right)^{r+s-1} T^{b_1\dots b_{r-1}\hat{\lambda}|\underline{\hat{\nu}}}$$

$$\mathrm{Ber}^*\left(\begin{array}{c|c:c} T^{\underline{a}(a_i \leftarrow b_j)|\underline{\hat{\mu}}} & T^{\underline{a}(a_i \leftarrow c)|\underline{\hat{\mu}}} & T^{\underline{a}(a_i \leftarrow \hat{\nu}_\beta)|\underline{\hat{\mu}}} \\ \hline T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow b_j)} & T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow c)} & T^{*\underline{a}|\underline{\hat{\mu}}(\hat{\mu}_\alpha \leftarrow \hat{\nu}_\beta)} \end{array}\right)_{\substack{i,j=1\dots r,\\ \alpha=1\dots s\\ \beta=1\dots s-1}}$$

$$= \left(T^{\underline{a}|\underline{\hat{\mu}}}\right)^{-r-s+1} T^{*b_1\dots b_r|c\hat{\nu}_1\dots\hat{\nu}_{s-1}}$$

Here $\underline{a} = a_1 \dots a_r$.

The dual case $Gr_{r|1}(n|1)$

$(Gr_{r-2|1}(n|1)$ is dual to $Gr_{2|0}(n|1)$.)

<u>Theorem [S'23]</u>

$$\theta^{\underline{a}c} := T^{*\underline{a}|c}\left(T^{\underline{a}|\hat{1}}\right)^2 \tag{2}$$

is anti-symmetric in all its indices $a_1, \ldots, a_r, c$.
Under a change of basis in L, $U \mapsto gU$, the variables $\theta^{\underline{a}c}$ transform as $\theta^{\underline{a}c} \mapsto (\operatorname{Ber} g)\, \theta^{\underline{a}c}$.

Theorem [S'23] For the super Grassmannian $Gr_{r|1}(n|1)$, one can introduce new homogeneous coordinates $P^{\underline{a}}, P^{*\underline{a}}, \theta^{\underline{a}c}$, antisymmetric in all indices, that are expressed in terms of the original super Plücker coordinates as

$$P^{\underline{a}} := T^{\underline{a}|\widehat{1}}, \tag{3}$$

$$P^{*\underline{a}} := T^{*\underline{a}|\widehat{1}}, \tag{4}$$

$$\theta^{\underline{a}c} := T^{*\underline{a}|c}(P^{\underline{a}})^2 . \tag{5}$$

In terms of these new coordinates, the super Plücker relations have the following form: $P^{*\underline{a}} = (P^{\underline{a}})^{-1}$, and

$$\sum_{i=1}^{r} P^{b_i a_2 \dots a_r} P^{\underline{b}(b_i \leftarrow a_1)} = P^{\underline{a}} P^{\underline{b}}, \tag{6}$$

$$-\sum_{i=1}^{r} \theta^{\underline{a} b_i} P^{\underline{b}(b_i \leftarrow c)} + \theta^{\underline{a}c} P^{\underline{b}} = P^{\underline{a}} \theta^{\underline{b}c}, \tag{7}$$

where $\underline{a}(a_i \leftarrow b_j) = a_1 \dots a_{i-1} b_j a_{i+1} \dots a_r$, and $\underline{b}(b_i \leftarrow c) = b_1 \dots b_{i-1} c b_{i+1} \dots b_r$.

Example

For the particular case of $Gr_{2|1}(n|1)$, we have the even Plücker relations

$$\det \begin{pmatrix} P^{a_1 b_1} & P^{a_1 b_2} \\ P^{a_2 b_1} & P^{a_2 b_2} \end{pmatrix} = P^{a_1 a_2} P^{b_1 b_2}\,, \tag{8}$$

and the odd Plücker relations

$$-\theta^{a_1 a_2 b_1} P^{c b_2} - \theta^{a_1 a_2 b_2} P^{b_1 c} + \theta^{a_1 a_2 c} P^{b_1 b_2} = \theta^{b_1 b_2 c} P^{a_1 a_2}\,. \tag{9}$$

Cluster algebras

Fomin and Zelevinsky (2002). A cluster algebra is a commutative algebra with a family of distinguished generators (the "cluster variables") grouped into overlapping subsets (the "clusters") of the same size, which are constructed recursively using "mutations".

Example. The homogeneous coordinate ring of $Gr_2(n)$, the quotient of the polynomial ring in $P^{ab}, 1 \leq a < b \leq n$ by $P^{ac}P^{bd} = P^{ab}P^{cd} + P^{ad}P^{bc}$.

For $n = 5$, the relations imply that to check the positivity of all ten P^{ab}, it is enough to check the positivity of only

$$\{P^{13}, P^{14}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}$$

or of only

$$\{P^{24}, P^{14}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}$$

or of only ...

So all ten P^{ab} are positive iff positive are all elements of just one of the following “clusters”:

$$\{P^{13}, P^{14}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}, \tag{10}$$

$$\{P^{24}, P^{14}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}, \tag{11}$$

$$\{P^{24}, P^{25}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}, \tag{12}$$

$$\{P^{35}, P^{25}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}, \tag{13}$$

$$\{P^{35}, P^{13}, P^{12}, P^{23}, P^{34}, P^{45}, P^{15}\}. \tag{14}$$

Every cluster has the same five "frozen cluster variables".

$$\{P^{13}, P^{14}, \underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}\}, \tag{15}$$
$$\{P^{24}, P^{14}, \underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}\}, \tag{16}$$
$$\{P^{24}, P^{25}, \underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}\}, \tag{17}$$
$$\{P^{35}, P^{25}, \underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}\}, \tag{18}$$
$$\{P^{35}, P^{13}, \underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}\}. \tag{19}$$

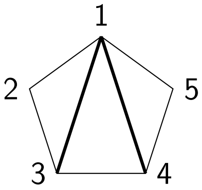
1. Only one variable is changing while going from a cluster to the next one.
2. This new variable can be expressed in terms of the "old" cluster via an expression of the form

$$P_{\text{new}} = \frac{P^{**}P^{**} + P^{**}P^{**}}{P_{\text{old}}}.$$

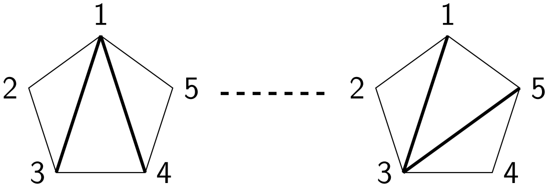
E.g. going from (15) to (16), it is

$$P^{24} = \frac{P^{12}P^{34} + P^{14}P^{23}}{P^{13}}.$$

These clusters and the mutation laws can be parametrized by particular triangulations of a regular n-gon.



Cluster variables P^{13}, P^{14}, ... are represented by proper diagonals. Frozen cluster variables $\underline{P^{12}, P^{23}, P^{34}, P^{45}, P^{15}}$ are represented by boundary edges.



Mutation: P^{14} mutates into P^{35}, which according to the Ptolemy theorem can be found from $P^{14}P^{35} = P^{13}P^{45} + P^{15}P^{34}$, which coincides with one of the Plücker relations.

Our super cluster structure

An interpretation of super Plücker relations appears to be a complicated task at the moment and possibly requires first some simplification of the relations.

So far we have a super cluster structure based on super Plücker relations for $Gr_{2|0}(n|1)$. Here, super Plücker relations are

$$T^{ac}T^{bd} = T^{ab}T^{cd} + T^{ad}T^{bc} \,, \tag{20}$$

$$T^{ab}\theta^{c} = T^{ac}\theta^{b} + T^{cb}\theta^{a} \,, \tag{21}$$

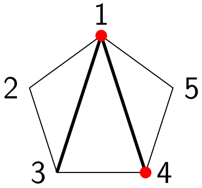
$$T^{ab}T^{\hat{1}\hat{1}} = -2\theta^{a}\theta^{b} \,, \tag{22}$$

$$\theta^{a}T^{\hat{1}\hat{1}} = 0 \,, \tag{23}$$

$$\left(T^{\hat{1}\hat{1}}\right)^{2} = 0 \,. \tag{24}$$

As every chart in $Gr_{2|0}(n|1)$ has some T^{ab} invertible, one can express the even nilpotent $T^{\hat{1}\hat{1}}$ in terms of the remaining variables. Hence we are left with only (20) and (21).

Decorated triangulations. Consider a regular n-gon and label its vertices counter clockwise. Decorate: exactly one of the proper diagonals has marked vertices (representing two odd cluster variables).



Every cluster has $n-3$ even and two odd variables (not counting frozen even cluster variables) where the indices are in strictly increasing order:

$$\{T^{ab}, \underbrace{T^{cd}, \ldots, T^{kl}}_{n-4} | \theta^a, \theta^b\}.$$

Odd mutation.

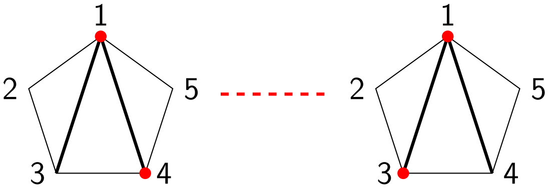
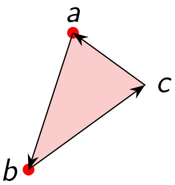


Figure: Example of an odd mutation: θ^4 mutates into θ^3.

θ^3 can be expressed in terms of the "old" cluster using an odd super Plücker relation. The signs are defined uniquely once the vertices of the n-gon are ordered.

$$\theta^3 = \frac{T^{13}\,\theta^4 + \theta^1\, T^{34}}{T^{14}}$$

An odd mutation means θ^a or θ^b mutates into θ^c in a triangle:



The linear order on the vertices of the n-gon implies that either $a < b < c$, or $b < c < a$, or $c < a < b$.
(The mutation formulas are the same for mutating θ^a or θ^b into θ^c.)
In anti-symmetric notation, the odd super Plücker relation, $T^{ab}\theta^c + T^{bc}\theta^a + T^{ca}\theta^b = 0$ implies $\theta^c = \frac{T^{cb}\theta^a + T^{ac}\theta^b}{T^{ab}}$. Then
1) for $a < b < c$ we have $\theta^c = \frac{-T^{bc}\theta^a + T^{ac}\theta^b}{T^{ab}}$;
2) for $b < c < a$ we have $\theta^c = \frac{-T^{bc}\theta^a - T^{ca}\theta^b}{-T^{ba}}$;
3) for $c < a < b$ we have $\theta^c = \frac{T^{cb}\theta^a - T^{ca}\theta^b}{T^{ab}}$.
So if c is "in the middle" then the sign is $+$.

Even mutation. A flip with marked points also moving:

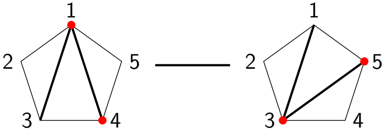


Figure: $\{T^{14}, \theta^1, \theta^4\}$ mutate into $\{T^{35}, \theta^3, \theta^5\}$.

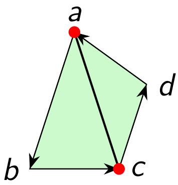
Variables $\{T^{35}, \theta^3, \theta^5\}$ can be expressed in terms of the "old" cluster by an even and two odd Plücker relations. The signs are uniquely defined by the order of the vertices.

$$T^{35} = \frac{T^{13}\, T^{45} + T^{15}\, T^{34}}{T^{14}}$$

$$\theta^3 = \frac{T^{13}\, \theta^4 + \theta^1\, T^{34}}{T^{14}}$$

$$\theta^5 = \frac{T^{15}\, \theta^4 - \theta^1\, T^{45}}{T^{14}}$$

So an even mutation means that the decorated diagonal ac mutates into decorated diagonal bd.



As the order on the n-gon counter clockwise we have the following cases:

- $a < b < c < d$ and so $\theta^b = \frac{\cdots + \cdots}{\cdots}$ and $\theta^d = \frac{\cdots - \cdots}{\cdots}$
- $b < c < d < a$ and so $\theta^b = \frac{\cdots - \cdots}{\cdots}$ and $\theta^d = \frac{\cdots + \cdots}{\cdots}$
- $c < d < a < b$ and so $\theta^b = \frac{\cdots - \cdots}{\cdots}$ and $\theta^d = \frac{\cdots + \cdots}{\cdots}$
- $d < a < b < c$ and so $\theta^b = \frac{\cdots + \cdots}{\cdots}$ and $\theta^d = \frac{\cdots - \cdots}{\cdots}$

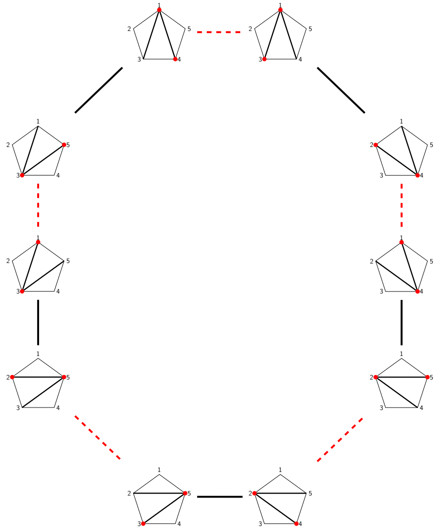
The even Plücker relation

$$T^{ac}T^{bd} = T^{ab}T^{cd} + T^{ad}T^{bc} \tag{25}$$

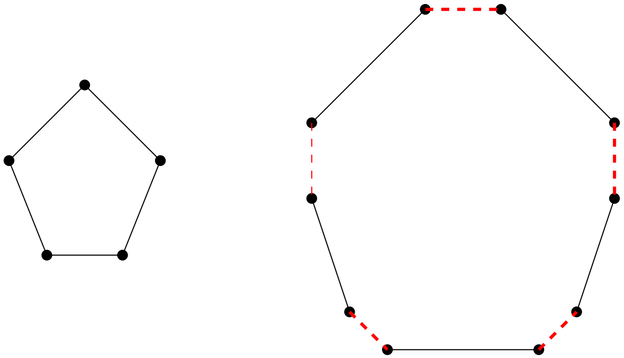
is invariant under cyclic permutations (and so the signs are always the same).

The exchange graph.

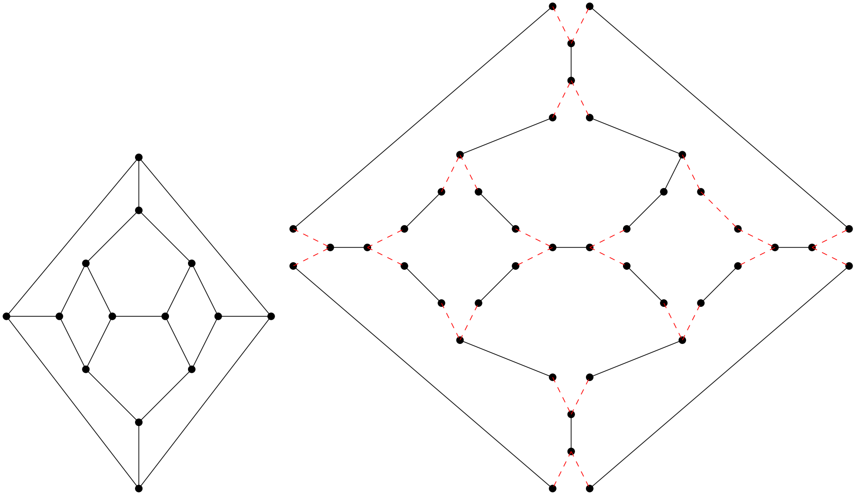
The exchange graph for $Gr_{2|0}(n|1)$ is a "blow-up" of the exchange graph for $Gr_{2|0}(n|0)$. E.g. for $Gr_{2|0}(5|1)$:



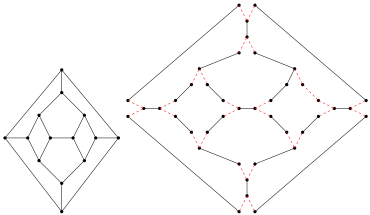
In general, one can show that the exchange graphs for the cluster structures corresponding to $Gr_{2|0}(n|1)$ can be obtained from the exchange graphs for $Gr_2(n)$, by replacing every vertex with a set of $n-3$ vertices that correspond to the same triangulation but different markings (so they are connected by odd mutations only). For example, the exchange graph for $Gr_{2|0}(5|1)$ on can be obtained from the exchange graph for $Gr_2(5)$ as follows:



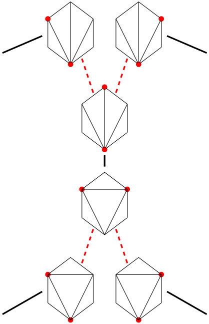
The exchange graph for $Gr_{2|0}(6|1)$ from that for $Gr_2(6)$:



The exchange graph for $Gr_{2|0}(6|1)$ from that for $Gr_2(6)$:



Zoom in:

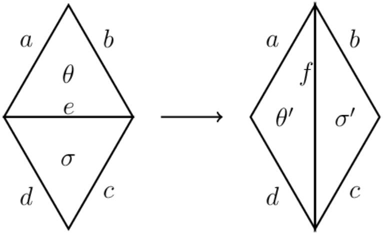


Modification of Penner-Zeitlin's super Ptolemy equations

An alternative approach for construction of super cluster algebras is based on Penner-Zeitlin's super Ptolemy equation [6] for the decorated super Teichmüller space $\widetilde{\mathcal{ST}}$.
This super Ptolemy equation is used as an exchange relation in the Musiker-Ovenhouse-Zhang [2] version of super cluster structure for $\widetilde{\mathcal{ST}}$.
Here we propose a re-writing of Penner-Zeitlin's super Ptolemy equation in a new form.

The classical *Teichmüller space* $\mathcal{T}$ is a moduli space for hyperbolic metrics on a (topological) surface F.
Penner's *decorated Teichmüller space* [5] $\widetilde{\mathcal{ST}}$ assign a number to each puncture that fixes the height of the horocycles. Then the "regularized lengths" ℓ for the geodesics with the endpoints at the punctures are the lengths of the truncated parts. Then Penner introduces λ-*lengths*, $\lambda = exp(\ell/2)$.
Penner equips the space $\widetilde{\mathcal{T}}$ with global coordinate systems that are enumerated by "ideal" triangulations of F, the vertices of which are the punctures. For a given triangulation, the λ-lengths of the edges are the coordinates. Transforming one triangulation into another by a flip leads to a changes of λ-lengths governed by the Ptolemy equation.

In 2019, Penner-Zeitlin [6] introduced global coordinates for the connected components of the *decorated super Teichmüller space* $\widetilde{\mathcal{ST}}$.



$$ef = (ac + bd)\left(1 + \frac{\sigma\theta\sqrt{Z}}{1+Z}\right), \tag{26}$$

$$\theta' = \frac{\theta + \sigma\sqrt{Z}}{\sqrt{1+Z}}, \tag{27}$$

$$\sigma' = \frac{\sigma - \theta\sqrt{Z}}{\sqrt{1+Z}}, \tag{28}$$

where $Z = ac/bd$.

$Z = ac/bd$

Lemma

Define new even variables:

$$\bar{e} := e\left(1 + \frac{\sigma\theta\sqrt{Z}}{1+Z}\right)^{1/2},$$
$$\bar{f} := f\left(1 + \frac{\sigma'\theta'\sqrt{Z}}{1+Z}\right)^{1/2}$$

(note $\sigma\theta = \sigma'\theta'$). Then the even super Ptolemy relation (26) *becomes*

$$\bar{e}\bar{f} = ac + bd\,. \tag{29}$$

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