The homotopy moment map for Chern–Simons field theory

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Homotopy moment maps

Let X be a manifold, with de Rham complex $\Omega^*(X)$. Let n be a natural number.

Definition

An n-multisymplectic form on X is a closed differential form $\omega \in \Omega^{n+1}(X)$ such that the morphism

$$\xi \in \mathsf{Vec}(X) \mapsto \iota(\xi)\omega \in \Omega^n(X)$$

is injective.

The case $n = 1$ of symplectic forms is special, since if X is finite-dimensional, the morphism $\xi \mapsto \iota(\xi)\omega$ is not just injective, but bijective.

Let $\mathfrak{h}$ be a Lie algebra with differential action on X:

$$\rho : \mathfrak{h} \to \mathsf{Vec}(X).$$

Let $C^*(\mathfrak{h})$ be the Chevalley–Eilenberg complex of $\mathfrak{h}$, with differential δ_0. Denote the tensor product of $C^*(\mathfrak{h})$ and $\Omega^*(X)$ by

$$C^*(\mathfrak{h}) \otimes \Omega^*(X).$$

Choose a basis $\{c_i\}$ of $\mathfrak{h}$, and dual basis $\{c^i\}$ of $\mathfrak{h}^*$; denote the structure coefficients of $\mathfrak{h}$ by A_{ij}^k. Form the operator

$$\iota = \sum_i c^i \otimes \iota(\rho_i) : C^p(\mathfrak{h}) \otimes \Omega^q(X) \to C^{p+1}(\mathfrak{h}) \otimes \Omega^{q-1}(X),$$

of total degree 0.

Definition (Callies, Frégier, Rogers, Zambon)

A **homotopy moment map** for the action ρ of $\mathfrak{h}$ on an n-multisymplectic manifold (X, ω) is an element $\mu \in C^*(\mathfrak{h}) \otimes \Omega^*(X)$ of total degree $n-1$ such that

$$(\delta_0 + d)\mu = e^{\iota}\omega - \omega.$$

The case of symplectic manifolds: $n = 1$

If $n = 1$, we have $\iota\omega = \sum_i c^i \iota(\rho_i)\omega \in \mathfrak{h}^* \otimes \Omega^1(X)$. Thus, a homotopy moment map is an element $\mu = \sum_i c^i \mu_i \in \mathfrak{h}^* \otimes \Omega^0(X)$ such that

$$d\mu_i = -\iota(\rho_i)\omega$$

and

$$\sum_k A_{ij}^k \mu_k = \iota(\rho_i)\iota(\rho_j)\omega = -\mathcal{L}(\rho_i)\mu_j = \{\mu_i, \mu_j\}.$$

In other words, μ is a moment map for the action ρ of $\mathfrak{h}$ on the symplectic manifold (X, ω).

Reformulation for exact n-multisymplectic manifolds

Definition

An n-multisymplectic manifold (X, ω) is **exact** if $\omega = d\alpha$, for $\alpha \in \Omega^n(X)$.

The Lie derivative $c_i \mapsto \mathcal{L}(\rho_i)$ makes the de Rham complex $\Omega^*(X)$ into a differential graded $\mathfrak{h}$-module. This motivates replacing the complex $C^*(\mathfrak{h}) \otimes \Omega^*(X)$ by the complex $C^*(\mathfrak{h}, \Omega^*(X))$, with differential $\delta + d$, where

$$\delta = \delta_0 + \sum_i c^i \mathcal{L}(\rho_i).$$

Lemma

$$\delta + d = e^{\iota} \circ (\delta_0 + d) \circ e^{-\iota}$$

The equation for a homotopy moment map becomes

$$(\delta_0 + d)(\alpha + \mu) = e^{\iota}\omega.$$

Define

$$\nu = e^{\iota}(\alpha + \mu) \in C^*(\mathfrak{h}, \Omega^*(X)),$$

or equivalently,

$$\mu = e^{-\iota}\nu - \alpha.$$

For exact n-multisymplectic manifolds, a homotopy moment map is an equivariant extension of α:

$$(\delta + d)\nu = \omega.$$

The variational bicomplex

Let $p : E \to M$ be a fiber bundle, with jet-space $J_\infty(p)$. An **adapted coordinate system** at a point $e \in E$ is a coordinate system $(t^1, \dots, t^n; u^1, \dots, u^N)$ such that the coordinates $(t^1, \dots, t^n)$ are pulled back from a coordinate system around $p(e) \in M$.

The coordinates t^μ are the **independent coordinates**, and M is the **worldsheet**; the coordinates u^a are the **dependent coordinates**, identified with the fields of the theory. Denote $\partial/\partial t^\mu$ by ∂_μ.

An adapted coordinate system gives rise to coordinates

$$(t^1, \dots, t^n; \partial^I u^1, \dots, \partial^I u^N)$$

on the jet-space $J_\infty(p)$, where I ranges over multi-indices $(i_1, \dots, i_n) \in \mathbb{N}^n$, and

$$\partial^I = \partial_1^{i_1} \dots \partial_n^{I_n}.$$

Denote partial differentiation $\partial/\partial(\partial^I u_i)$ with respect to a jet coordinate by $\partial_{i,I}$.

Let $\mathcal{O}_\infty$ be sheaf of algebras over M whose sections are smooth functions in the coordinates $(t^1, \dots, t^n; \partial^I u^1, \dots, \partial^I u^N)$.

The variation de Rham complex

The **variational de Rham complex** Ω^*_∞ is the de Rham complex associated to $\mathcal{O}_\infty$.

It is bigraded: the differentials dt^μ have bidegree $(1,0)$, and the differentials

$$\theta^i_I = d(\partial^I u^i) - \sum_\mu \partial_\mu \partial^I u^i \, dt^\mu$$

have bidegree $(0,1)$.

The differential d on Ω_∞ breaks into horizontal and vertical parts

$$d^{1,0} = \sum_\mu dt^\mu D_\mu : \Omega^{p,q}_\infty \longrightarrow \Omega^{p+1,q}_\infty, \qquad d^{0,1} = \sum_{i,I} \theta^a_I \, \partial_{i,I} : \Omega^{p,q}_\infty \longrightarrow \Omega^{p,q+1}_\infty.$$

where

$$D_\mu = \partial_\mu + \sum_{i,I} \partial_\mu \big(\partial^I u^i \big) \, \partial_{i,I}.$$

The variational n-multisymplectic form of a classical field theory

A first-order Lagrangian density L determines a classical field theory. In an adapated coordinate system,

$$L = L(t, u_i, \partial_\mu u_i)\, dt^1 \wedge \ldots \wedge dt^n \in \Omega_\infty^{n,0}.$$

From L, we construct a Lepage form

$$\alpha = L - \partial_{i,\mu} L\, \theta_i \wedge \iota(\partial_\mu)\, dt^1 \wedge \ldots \wedge dt^n \in \Omega_\infty^{n,0} \oplus \Omega_\infty^{n-1,1} \subset \Omega_\infty^n.$$

The differential $\omega = d\alpha \in \Omega_\infty^{n+1}$ extends Noether's symplectic form off-shell:

$$\omega = \frac{\delta L}{\delta u_i}\, du_i \wedge dt^1 \wedge \ldots \wedge dt^n + \cdots \in \Omega_\infty^{n,1} + \Omega_\infty^{n-1,2}.$$

Modulo mild conditions on L which guarantee nondegeneracy, ω is a variational analogue of a n-multisymplectic form.

Variational Cartan calculus

Derivations $\xi \in \mathsf{Der}(\mathcal{O}_\infty)$ of $\mathcal{O}_\infty$ are the variational vector fields. There is a variational Cartan calculus: to a variational vector field ξ, we associate the contraction

$$\iota(\xi) : \Omega^*_\infty \to \Omega^{*-1}_\infty$$

and the Lie derivative

$$\mathcal{L}(\xi) = d \circ \iota(\xi) + \iota(\xi) \circ d : \Omega^*_\infty \to \Omega^*_\infty.$$

Fix a fiber bundle $p : E \longrightarrow B$ and a (first-order) Lagrangian density $L \in \Omega^{n,0}_\infty$ with associated Lepage form $\alpha \in \Omega^n_\infty$ and variational n-multisymplectic form $\omega \in \Omega^{n+1}_\infty$.

Consider a Lie algebra $\mathfrak{h}$ and a variational action of $\mathfrak{h}$

$$\rho : \mathfrak{h} \to \mathsf{Der}(\mathcal{O}_\infty)$$

such that $\mathcal{L}(\rho)\omega = 0$.

Definition

A **variational homotopy moment map** for ρ is an element $\nu \in C^*(\mathfrak{h}, \Omega^*_\infty)$ of total degree $n-1$ such that

$$(\delta + d)\nu = \omega.$$

The Chern–Simons classical field theory

Let M be a closed oriented 3-manifold. Let $\mathfrak{g}$ be a reductive Lie algebra, with invariant inner product $(-,-)$. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$.

Let P be a G-principal bundle over M. We consider a field theory over M, where the fields (dependent coordinates) are the components of a connection over P.

Over a chart $U \subset M$ where P is trivialized, the Lepage form of the Chern–Simons Lagrangian theory equals

$$\alpha = \tfrac{1}{2}(A, dA) + \tfrac{1}{6}(A, [A, A]).$$

If G is simply connected, then P is a trivial bundle, and the Lepage form is globally defined. In general, P is not trivial, and the Lepage form is a Čech cochain $\alpha \in \check{C}^*(\mathcal{U}, \Omega^*_\infty)$ of total degree 3.

The 3-multisymplectic form of Chern–Simons field theory

The 3-multisymplectic form equals

$$\omega = \tfrac{1}{2}(\mathbb{F}, \mathbb{F}) = (F, d^{0,1}A) + \tfrac{1}{2}(d^{0,1}A, d^{0,1}A),$$

where $F = d^{1,0}A + \tfrac{1}{2}[A, A]$ is the curvature, and

$$\mathbb{F} = dA + \tfrac{1}{2}[A, A] = d^{0,1}A + F$$

is its analogue in the variational bicomplex.

From this point, the slides from the talk contained an error, which is corrected here: the multisymplectic form of Chern–Simons theory is only nondegenerate on evolutionary vector fields.

This 3-form is nondegenerate on **evolutionary** vector fields: those of the form

$$\mathrm{pr}\left\langle X, \frac{\partial}{\partial A} \right\rangle = \sum_I \left\langle \partial^I X, \frac{\partial}{\partial(\partial^I A)} \right\rangle,$$

where $X \in \Omega^{1,0}_\infty \otimes \mathfrak{g}$.

The Atiyah Lie algebra

The Atiyah Lie algebra $\mathcal{A}(P)$ of a principal bundle P is spanned by covariant derivatives ∇^A_ξ, and gauge transformations $\eta \in P \times_G \mathfrak{g}$.

In adapted coordinates associated to a local trivialization of P over an open subset $U \subset M$, we have $\nabla^A_\xi + \eta = \xi^\mu \partial_\mu + \mathrm{ad}\big(\iota(\xi)A + \eta\big)$.

The evolutionary vector fields

$$\begin{aligned}\rho\big(\nabla^A_\xi + \eta\big) &= \mathrm{pr}\left\langle -\mathcal{L}(\xi)A + \mathrm{ad}\,\nabla^A\big(\iota(\xi)A + \eta\big), \frac{\partial}{\partial A}\right\rangle \\ &= \mathrm{pr}\left\langle -\iota(\xi)\mathbb{F} + \mathrm{ad}\,\nabla^A\eta, \frac{\partial}{\partial A}\right\rangle\end{aligned}$$

yield an action of the Atiyah Lie algebra on the variational de Rham complex.

The (local) homotopy moment map

We have $\iota\mathbb{F} = \nabla^A\eta$, and hence $e^\iota\mathbb{F} = \mathbb{F} + \nabla^A\eta$

Since $\delta_0 A = 0$, we see that $(\delta_0 + d)A + \frac{1}{2}[A, A] = \mathbb{F}$.

Applying e^ι to both sides, and using $e^\iota A = A$, we see that

$$(\delta + d)A + \frac{1}{2}[A, A] = e^\iota\mathbb{F}.$$

Hence

$$(\delta + d)(A - \eta) + \frac{1}{2}[A - \eta, A - \eta] = \mathbb{F}.$$

Theorem

$$\nu = \frac{1}{2}(A - \eta, (\delta + d)(A - \eta)) + \frac{1}{6}(A - \eta, [A - \eta, A - \eta])$$

is a variational homotopy moment map for the Chern–Simons theory.

The (global) homotopy moment map

The point of this talk is that when P is not a trivial bundle, the Chern–Simons action, and Lepage form, are Čech cochains for a cover $\mathcal{U}$ of M.

This forces the variational homotopy moment map

$$\nu \in C^*(\mathfrak{g}, \check{C}^*(\mathcal{U}, \Omega^*_\infty))$$

to be a Čech cochain of total degree 3, satisfying the equation

$$(\delta + \check{\delta} + d)\nu = \omega.$$

The formulas for the higher degree Čech cochains components of ν are left as an exercise.

Since $\mu = e^{-\iota}\nu - \alpha$ is not an element of $C^*(\mathfrak{g}, \Omega^*_\infty)$, we have an example of a homotopy moment map that is a differential character, but not a globally defined differential form.

This phenomenon only arises because $n > 1$.