Math 255 **Test #2** **Due 5:00 pm, Thursday, December 22, 2011**

This is a take-home examination. You may use class notes, homework, handouts, the text, any multivariable calculus text, the web site mentioned in Part III, and nothing else. Don't discuss the exam with other class members, but you should feel free to consult with me about any questions you might have. Hints are available on some of the problems.

If you no longer have the text you used for multivariable calculus, you can use Stewart's *Multivariable Calculus*, which is on reserve in the Keefe Science Library in the Merrill Science Building.

Part I. The Tractrix

1. [15 Points] In class, we showed that the tractrix $y = f(x)$, $0 < x \leq 1$, can be parametrized by

$$\alpha(t) = (\sin t, -\ln(\csc t + \cot t) + \cos t)$$

for $t \in (0, \pi/2]$.

a. Show that the function $f(x)$ that gives the tractrix has the formula

$$f(x) = \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}.$$

b. Show that the tractix can be parametrized by

$$\beta(t) = \left(\frac{1}{t}, \ln\left(t - \sqrt{t^2-1}\right) + \sqrt{1-\frac{1}{t^2}}\right)$$

for $t \geq 1$.

Part II. Curvature

2. [15 Points] Let $f(x, y)$ be a smooth function defined on $\mathcal{U} \subset \mathbb{R}^2$. Its graph $z = f(x, y)$ is a surface in $\mathbb{R}^3$ which can be parametrized by

$$X(x, y) = (x, y, f(x, y))$$

for $(x, y) \in \mathcal{U}$. Show that the curvature of this surface is given by the formula

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

3. [15 Points] Suppose that a smooth function $f(x, y)$ has a critical point at (x_0, y_0), meaning that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

a. If the graph has positive curvature at $(x_0, y_0, f(x_0, y_0))$, then prove that $f(x, y)$ has a local min or local max at (x_0, y_0).

b. If the graph has negative curvature at $(x_0, y_0, f(x_0, y_0))$, then prove that $f(x, y)$ has a saddle point at (x_0, y_0).

c. If the graph has zero curvature at $(x_0, y_0, f(x_0, y_0))$, then what can you conclude?

Hint: Look up the second derivative test for functions of two variables.

4. [15 Points] In class, we showed that the rotating disk gives the metric

$$dr^2 + \frac{r^2 d\theta^2}{1 - r^2},$$

where $0 \le \theta \le 2\pi$ and $0 \le r < 1$. Show that this metric has curvature $K = \dfrac{-3}{(1-r^2)^2}$. Thus $K < 0$. Since K is not constant, this is not a model of non-Euclidean geometry.

Part III. The Beltrami or Klein Disk Model of Non-Euclidean Geometry

The Poincaré disk model of non-Euclidean geometry uses the unit disk $D = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, and its "lines" consists of diameters of the circle together with arcs of circles that meet the boundary $u^2 + v^2 = 1$ at right angles. In this model, "angles" are just the usual Euclidean angles.

One model of non-Euclidean geometry we haven't discussed is the Beltrami or Klein disk model. Like the Poincaré disk model, this model uses the open unit disk $D = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, but its "lines" are really nice: they are line segments cut off by the intersection of straight lines (in the usual Euclidean sense) with D. The bad aspect of this model is that "angles" aren't the same as Euclidean angles. The web site `http://www.amherst.edu/~dacox/math23/` has a picture of these models (plus the Poincaré upper half plane model).

From the point of view of differential geometry, the Poincaré model uses D with the metric

$$\frac{4}{(1 - u^2 - v^2)^2}\big(du^2 + dv^2\big).$$

The Beltrami model, on the other hand, uses D with the *Beltrami metric*

$$ds^2 = \frac{1}{(1 - u^2 - v^2)^2}\big((1 - v^2)du^2 + 2uv\, du dv + (1 - u^2)dv^2\big).$$

5. [10 Points] Assume that the Beltrami model satisfies the axioms of neutral geometry

a. Give an example to show that parallels are not unique. Your example should include both asymptotic parallels and lines that are parallel but not asymptotically parallel. This shows that the Beltrami model gives non-Euclidean geometry.

b. Draw an asymptotic pencil in this model.

6. [30 Points] In the Euclidean 3-space $\mathbb{R}^3$, let the variables be x, y, t and consider the *Lorentz dot product*, which is defined by

$$(a, b, c) \cdot_L (d, e, f) = ad + be - cf.$$

The minus sign might look a bit strange, but this "dot product" is very useful in special relativity. The cone $x^2 + y^2 = t^2$ is called the *light cone* and consists of points (x, y, t) which left the origin at time $t = 0$ and are traveling at the speed of light (assuming units are chosen so that the speed of light is $c = 1$). Inside the light cone is the hyperboloid $x^2 + y^2 - t^2 = -1$. Most multivariable calculus books contain a picture of this surface.

This hyperboloid has two sheets, and we will denote the upper sheet by H^+ (since the z coordinate is positive on H^+). In this problem, you will show that the Lorentz dot product $\cdot_L$ gives a model of non-Euclidean geometry on H^+. Also let $D = \{(u, v) : u^2 + v^2 < 1\}$.

a. Take a point $(u, v, 1) \in \mathbb{R}^3$ with $u^2 + v^2 < 1$. Show that the ray from the origin through $(u, v, 1)$ intersects H^+ at the point

$$X(u, v) = \left(\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, \frac{1}{\sqrt{1 - u^2 - v^2}} \right).$$

Also show that $X : D \to H^+$ is one-to-one and onto, and draw a picture to illustrate your argument. Your picture should include a copy of D drawn at height $z = 1$ (see me if you are unsure what this means).

b. Show that $X(u, v) : D \to H^+$ gives a parametrization of the surface H^+.

c. The Lorentz dot product $\cdot_L$ on $\mathbb{R}^3$ gives a "dot product" on each tangent space $T_p(H^+)$ for $p \in H^+$. Although $\cdot_L$ behaves badly on $\mathbb{R}^3$ (vectors of zero length, etc.), the remarkable fact is that it gives the Beltrami metric when restricted to H^+. To see this, take $q \in D$ and let $p = X(q)$. Then, if we take $\vec{v}, \vec{w} \in T_p(H^+)$ and write

$$\vec{v} = aX_u(q) + bX_v(q)$$
$$\vec{w} = cX_u(q) + dX_v(q).$$

Then, using $\cdot_L$ on the tangent space leads to

$$\vec{v} \cdot_L \vec{w} = \big(aX_u(q) + bX_v(q)\big) \cdot_L \big(cX_u(q) + dX_v(q)\big).$$

If you represent this as a metric using du, dv in the usual way, show that you get the Beltrami metric defined on page 2 of the exam. Also verify that it is an abstract metric.

d. For the Beltrami disk model of non-Euclidean geometry on D, "lines" are given by $D \cap \ell$, where ℓ is a Euclidean straight line. Going back to the picture you drew of the map between D and H^+, what "lines" do you get on H^+? How does this compare to how great circles on the sphere are described?