Math 225: Chaos and Fractals
Final Exam, December 16, 2014

Name: ____

<u>INSTRUCTIONS</u>:

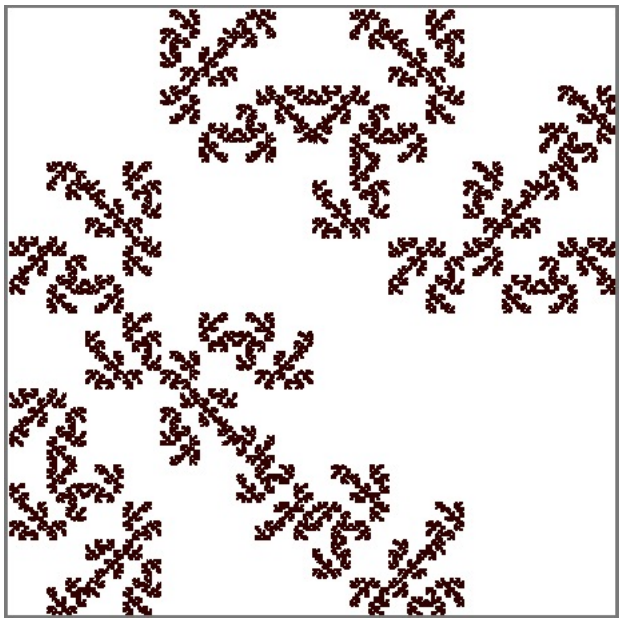
- Please answer ALL questions.
- You must show all of your work, & justify clearly where appropriate.
- Cite any theorem or result that you use.
- No books, notes, calculators, computers or cell phones allowed.

Good luck!

Please do not write in this table - for scoring purposes only.

Problem 1		(15)
Problem 2		(15)
Problem 3		(20)
Problem 4		(10)
Problem 5		(10)
Problem 6		(15)
Problem 7		(15)
Problem 8		(15)
Problem 9		(20)
TOTAL		(out of 135)

Problem 1(a). Find the IFS rules to generate the following fractals. For reference, the unit square is drawn in gray around each fractal. (You may not need to use all of the rows provided in the tables.)



Please list your *angles in radians*

r	s	θ	φ	e	f

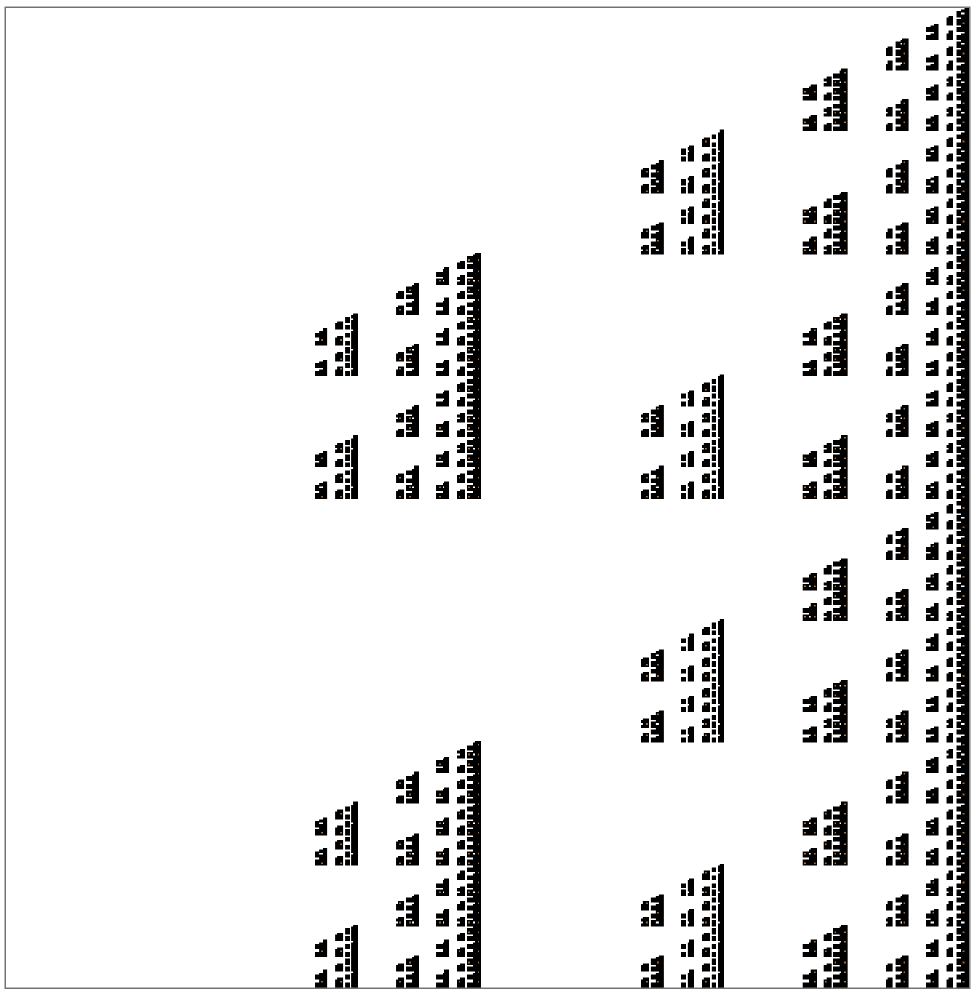
Problem 1(b). Compute the similarity dimension d_s of the fractal in part (a). *Your answer should be exact, and not a decimal approximation.*

Problem 2. The image $\mathcal{A}$ below is the attractor of a 1-IFS generated using the standard four contraction maps

$$T_1\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad T_2\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$T_3\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \qquad T_4\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

For reference, the unit square is drawn around the attractor:

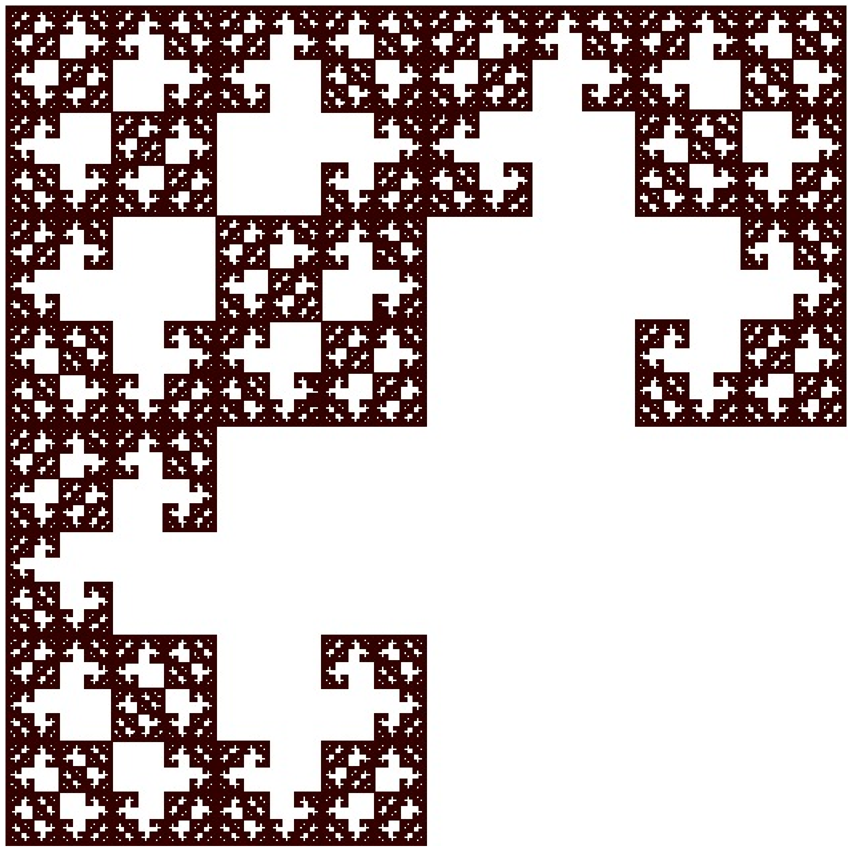


Problem 2 begins on the next page.

Problem 2(a). Draw the transition map for the 1-IFS that generates $\mathcal{A}$.

Problem 2(b). Can this 1-IFS be reduced to a 0-IFS? Justify your answer. If so, provide a minimal set of transformations (in terms of T_1, T_2, T_3, and T_4) that describe $\mathcal{A}$.

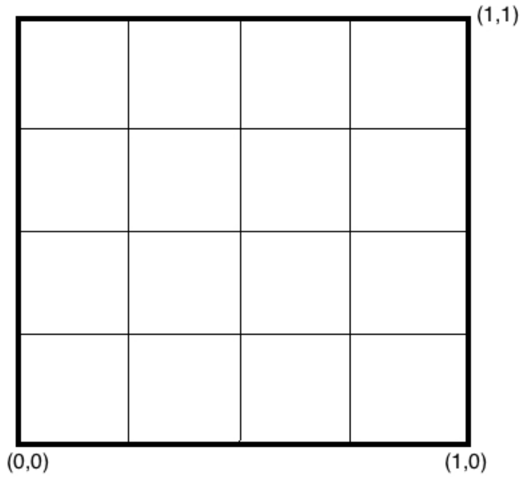
Problem 3(a). Give a set of IFS rules that generate the following attractor $\mathcal{B}$ (which as usual sits inside the unit square):



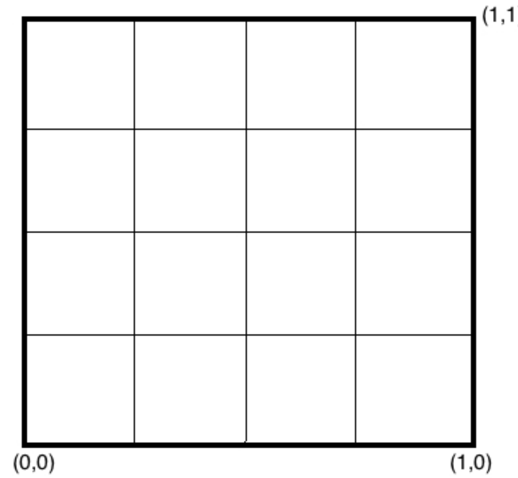
r	s	θ	φ	e	f

Please list your angles in radians. You may not need all rows provided.

Problem 3(b). Let S be the filled in unit square. If $\mathcal{T}$ denotes the collage map determined by your IFS from Problem 3(a), sketch $\mathcal{T}(S)$ and $\mathcal{T}^2(S)$ in the grids provided:



$\mathcal{T}(S)$



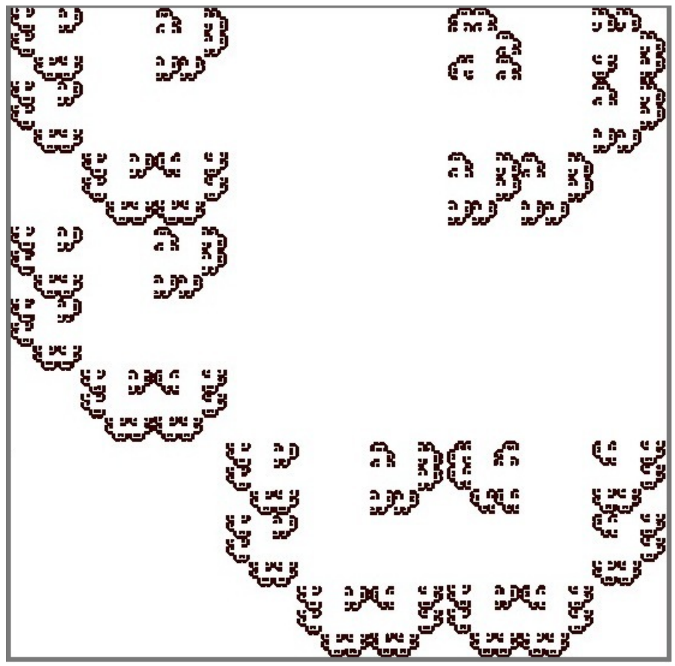
$\mathcal{T}^2(S)$

Problem 3(c). Compute the Hausdorff distances $h(S, \mathcal{B})$, $h(\mathcal{T}(S), \mathcal{B})$, and more generally, $h(\mathcal{T}^n(S), \mathcal{B})$ for any $n \geq 0$.

Problem 3(d). Deduce that the sequence $S, \mathcal{T}(S), \mathcal{T}^2(S), \mathcal{T}^3(S), \ldots$ converges to the fractal $\mathcal{B}$ in the Hausdorff metric.

Problem 4. What is the area of the fractal shown below?

As usual, the fractal is enclosed by the unit square for reference.



Problem 5(a). Define the s-dimensional Hausdorff measure $\mathcal{H}^s(A)$ of a set $A \subseteq \mathbb{R}^2$.

Problem 5(b). Define the Hausdorff dimension $d_h(A)$ of a set $A \subseteq \mathbb{R}^2$.

Problem 6. Use the definition of Hausdorff dimension (from Problem 5) to show that $\log(5)/\log(3)$ is an upper bound for the Hausdorff dimension of the fractal from Problem 4. *Do not use Hutchinson's theorem.*

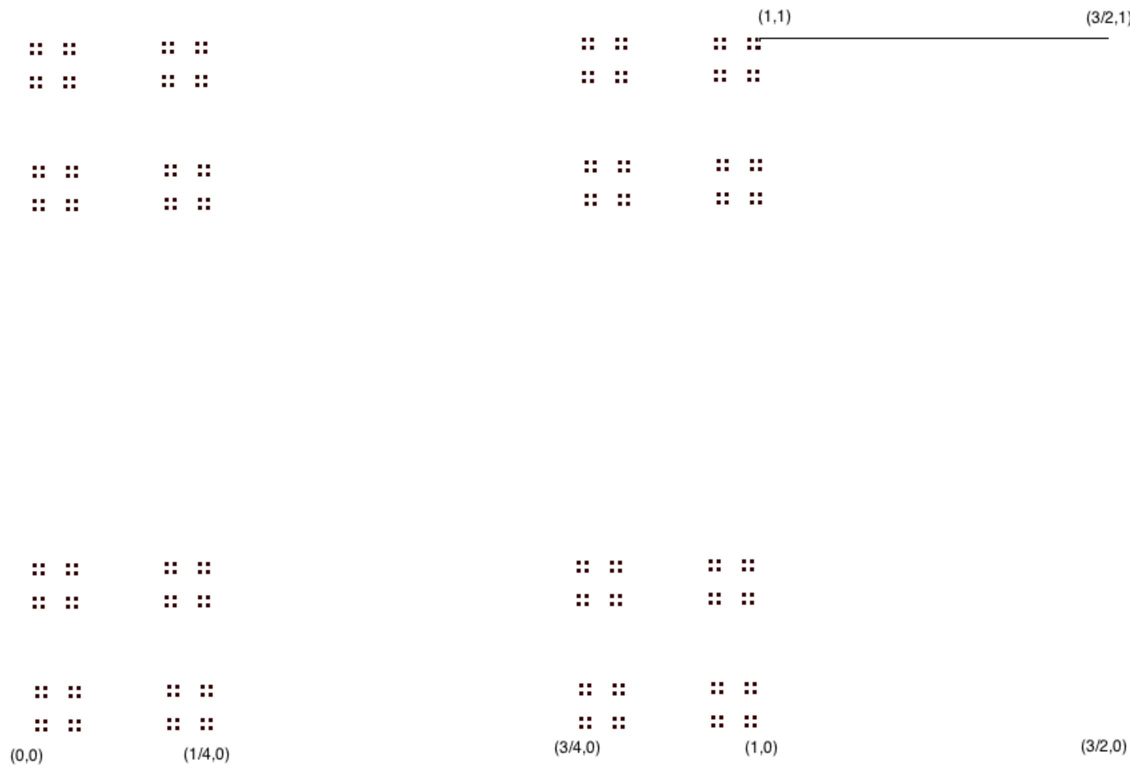
Problem 7. Let

$\mathcal{C} :=$ the "middle halves" Cantor set,

$\mathcal{L} :=$ the horizontal line segment from $(1,1)$ to $(3/2,1)$.

Specifically, $\mathcal{C}$ is the fractal in $\mathbb{R}$ formed by removing successive halves of intervals, and is given by the IFS $\{t_1, t_2\}$, where $t_1(x) = \frac{1}{4}x$, and $t_2(x) = \frac{1}{4}x + \frac{3}{4}$. (Here, $x \in \mathbb{R}$.)

The image below depicts $(\mathcal{C} \times \mathcal{C}) \cup \mathcal{L}$.



Problem 7 continues on the next page.

Problem 7(a). What is the definition of the box-counting dimension of a set $\mathcal{A} \subseteq \mathbb{R}^2$?

Problem 7(b). Let $\mathcal{F} = (\mathcal{C} \times \mathcal{C}) \cup \mathcal{L}$ as defined at the start of this problem. By constructing explicit box-coverings for $\mathcal{F}$, determine the box counting dimension $d_b(\mathcal{F})$.

Problem 8. In addition to the sets $\mathcal{C}$ and $\mathcal{L}$ from Problem 7, we define the sets $\mathcal{D}$ and $\mathcal{E}$ below.

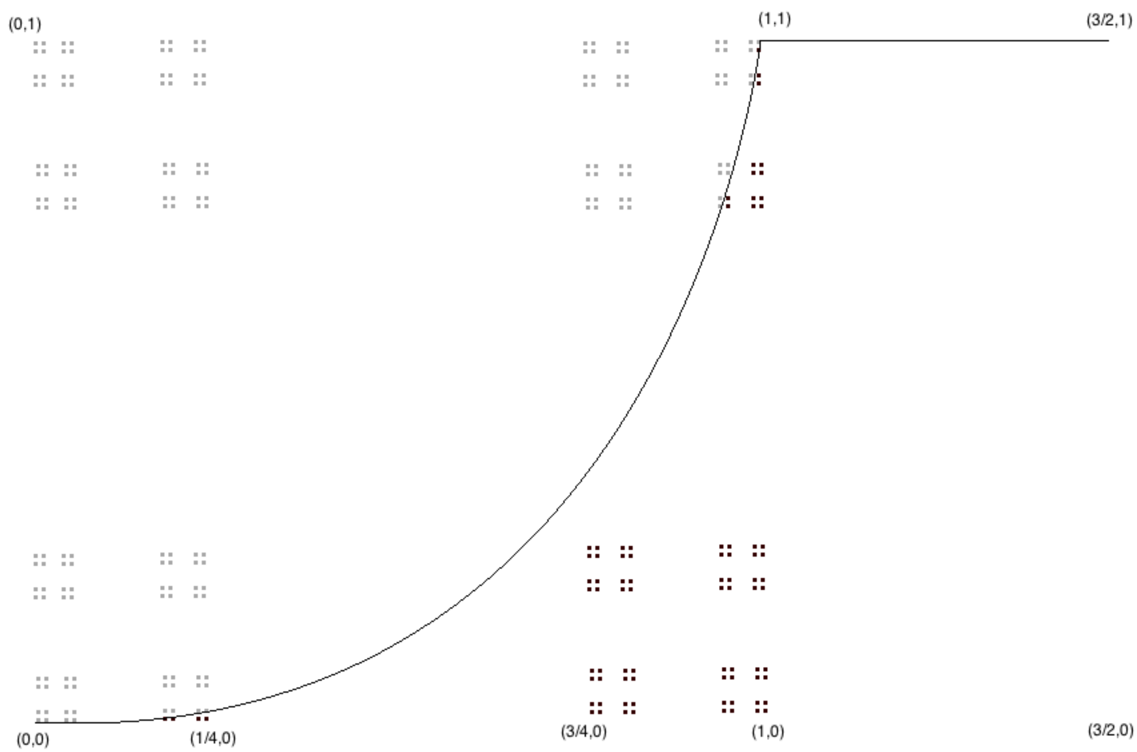
$\mathcal{C} :=$ the "middle halves" Cantor set,

$\mathcal{D} :=$ the portion of $\mathcal{C} \times \mathcal{C}$ lying below the line $y = x^3$,

$\mathcal{L} :=$ the horizontal line segment from $(1, 1)$ to $(3/2, 1)$,

$\mathcal{E} := \mathcal{D} \cup \mathcal{L}$.

These sets are depicted below, with various points marked for reference.



Problem 8 begins on the next page.

Problem 8. Compute the Hausdorff dimension $d_h(\mathcal{E})$ of $\mathcal{E}$. *You may use any method you choose, but must justify your answer.*

Problem 9(a). Let $F_c(z)$ be the complex function defined by $F_c(z) := z^2 + c$, where c is some complex number. Define the Julia set $J(F_c)$.

Problem 9(b). Let $c = 1$ and consider now $F_1(z) = z^2 + 1$. Show that if x is any real number (i.e. $x \in \mathbb{R}$), then $F_1(x) > x$.

Problem 9(c) continues on the next page.

Problem 9(c). Deduce that any real number x is NOT in the Julia set $J(F_1)$.

Problem 9(d) continues on the next page.

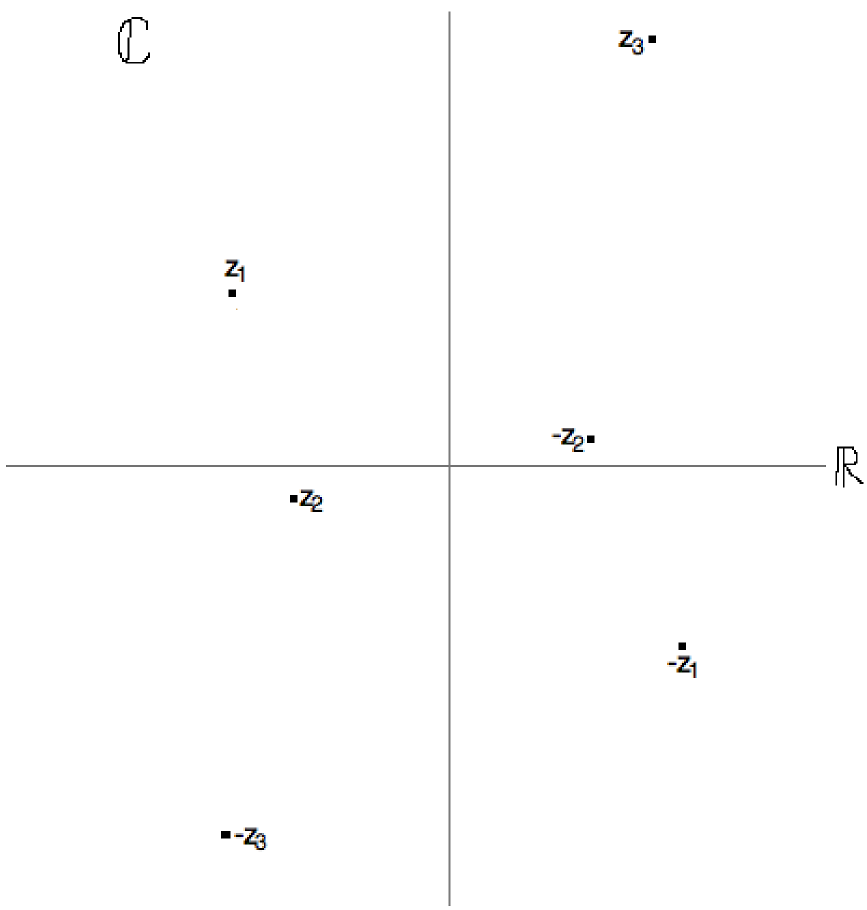
Problem 9(d). Notice by definition, we have that

$$F_1(z) = F_1(-z) = z^2 + 1.$$

This leads to the following fact.

FACT. The Julia set $J(F_1)$ is symmetric about the origin. That is, z is in $J(F_1)$ if and only if $-z$ is in $J(F_1)$.

Here is an image of some complex numbers $\pm z_1, \pm z_2, \pm z_3$ in $\mathbb{C}$ (with the real line $\mathbb{R}$ shown as well):



Problem 9(d) continues on the next page.

Problem 9(d). Using 9(c), the FACT from page 20, and the fact that $J(F_1) \neq \emptyset$, deduce that

(i) the Julia set $J(F_1)$ is disconnected, and

(ii) $c=1$ is not in the Mandelbrot set.

Scratch paper:

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