Math 211 Multivariable Calculus Final Exam
Sunday December 20, 2015

You have 3 hours for this exam. You may not use books, notes, calculators, cell phones or any other aids. Please turn off all electronic devices, including cell phones.

Explain your answers fully, showing all work for full credit. Answers involving sine, cosine or tangent of angles that are multiples of π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ should be evaluated exactly. Follow directions carefully: If the question tells you to use a particular method or definition, then you must use it in order to get credit. The total number of points is 100.

1. (10 points) Evaluate the integral by first switching the order of integration.

$$\int_0^1 \int_x^1 e^{x/y} dy dx$$

2. (10 points) The trajectories of two particles are given by

$$\vec{r_1}(t) = \langle t, t^2, t^3 \rangle$$

and

$$\vec{r_2}(t) = \langle -1 + 3t, 1 + 3t, -1 + 9t \rangle.$$

(a) Do the two particle collide? Explain.

(b) Find all the intersection points of the two paths.

3. (10 points) We are given two planes $x - z = 1$ and $y + 2z = 3$.

   (a) Find the acute angle between these two planes. Express it as an inverse cosine.

   (b) Find the equation of the plane that passes through the line of intersection of the two given planes and that is perpendicular to the plane $x + y - 2z = 1$.

4. (10 points) Let $f(x,y) = \begin{cases} \dfrac{9x^2 - 5xy + 3y^2}{3x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$.

(b) Prove that f is **not** continuous at $(0,0)$.

5. (10 points) Determine the absolute minimum and maximum values of

$$f(x, y) = x^2 + xy + y^2$$

on the disk $D = \{(x, y) \,|\, x^2 + y^2 \leq 4\}$ and the points where they occur.

6. (10 points) Calculate the volume of the solid region that is above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$ using triple integrals in

   (a) cylindrical coordinates;

   (b) spherical coordinates.

   For full credit, write out each triple integral and then evaluate.

7. (15 points) Let $f(x, y) = \ln(xy)$.

(a) Describe the domain of the function $f(x, y)$, and sketch the level curves at the levels $c = -1, 0, 1$.

(b) Find the direction of greatest rate of increase of the function at the point $(\frac{1}{2}, 1)$.

(c) Find the directions for which the directional derivative of f at the point $(\frac{1}{2}, 1)$ equals zero. Express as unit vectors.

(d) Suppose $x = g(s, t)$, $y = h(s, t)$, where g and h are differentiable functions. Find $\frac{\partial f}{\partial t}$ at $(s, t) = (1, 2)$ given the following information. (Note that you do not need to use *all* of the information given below.)

$$
\begin{aligned}
g(1,2) &= 4 & \qquad h(1,2) &= 2 \\
g_s(1,2) &= 0 & \qquad h_s(1,2) &= -1 \\
g_t(1,2) &= 3 & \qquad h_t(1,2) &= 5
\end{aligned}
$$

8. (10 points) Find the equation of the tangent plane to the surface

$$xe^{z+1} + yx^2 - \frac{z}{y} = 3$$

at the point $(-2, 1, -1)$. Express in form $ax + by + cz = d$.

9. (5 points) Calculate the surface area of the portion of the plane $x + y + z = a$ cut out by the cylinder $x^2 + y^2 = a^2$.

10. (10 points) Let C be the curve in the plane with parametrization $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \leq t \leq 2\pi$. Consider the line integral

$$\int_C 3x^2 y\, dx + x^3\, dy.$$

(a) Calculate this line integral using the Fundamental Theorem of Line Integrals.

(b) Calculate this line integral using Green's Theorem.

Extra credit (5 points): Prove using chain rule that the gradient $\nabla f(x, y)$ of a function $f(x, y)$ is perpendicular to the level curve of f that passes through (x, y).