# Math 211: Multivariable Calculus
# Final Exam

December 20, 2015

Attempt problems 1-11. Problem 12 is optional.
**Show all of your work.** Circle or box your final answer.
No notes, textbooks, calculators or outside help may be used on this exam.

1. (10 points) The velocity of a particle moving through space is given by the vector function
$$\vec{v}(t) = \left\langle \frac{t^3}{4}, e^{(t^2-2t)}, \sin\left(\frac{\pi t}{2}\right) \right\rangle.$$
Find the tangential and normal components of the acceleration at $t = 2$.

2. (10 points) Consider two planes: $P_1$ given by $x - z = 1$, and $P_2$ given by $y + 2z = 3$. Find the equation of the plane $\mathcal{P}$ that satisfies both of the following conditions:

   i. $\mathcal{P}$ passes through the line of intersection of $P_1$ and $P_2$

   ii. $\mathcal{P}$ is perpendicular to the plane $x + y - 2z = 1$.

3. (10 points) Let $f(x,y) = \begin{cases} \dfrac{9x^2 - 5xy + 3y^2}{3x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0). \end{cases}$

   (a) Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) Prove that $f$ is **not** continuous at $(0,0)$.

4. (10 points) Determine the absolute minimum and maximum values of
$$f(x,y) = x^2 + xy + y^2$$
on the disk $D = \{(x,y) \,|\, x^2 + y^2 \leq 4\}$ and the points where they occur.

5. (10 points) We wish to find the mass of the region $\mathcal{E}$ lying above the cone $z^2 = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 2$, whose density is given by the function $f(x,y,z) = 10 - z$. Set up (but **do *not* compute**) integrals to compute the mass of $\mathcal{E}$ using:

   (a) cylindrical coordinates.

   (b) spherical coordinates.

6. (5 points) Evaluate the integral by switching the order of integration.

$$\int_0^1 \int_x^1 e^{x/y} dy dx$$

7. (10 points) Let $f(x, y) = \ln(xy)$.

   (a) Sketch the domain of the function $f(x, y)$, and draw a contour map showing the level curves at the levels $c = -1, 0, 1$.

   (b) Find the direction of greatest rate of increase of the function at the point $Q = (\frac{1}{2}, 2)$.

   (c) In what direction from the point $Q = (\frac{1}{2}, 2)$ is the directional derivative equal to 0?

   (d) Suppose $x = g(s, t), y = h(s, t)$, where $g$ and $h$ are differentiable functions. Find $\frac{\partial f}{\partial t}$ at $(s, t) = (1, 2)$ given the following information. (Note that you do not need to use *all* of the information given below.)

$$\begin{aligned} g(1,2) &= \ 4 \\ g_s(1,2) &= \ 0 \\ g_t(1,2) &= \ 3 \end{aligned} \qquad\qquad \begin{aligned} h(1,2) &= \ 2 \\ h_s(1,2) &= -1 \\ h_t(1,2) &= \ 5 \end{aligned}$$

8. (5 points) Use a linear approximation to estimate $f(4.2, 1.9, 0.05)$ where

$$f(x, y, z) = \frac{x \cos z}{y^2}.$$

9. (a) (5 points) Let $C$ be the curve in the plane with parametrization $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \leq t \leq \pi/4$. Calculate the following line integral using the Fundamental Theorem of Line Integrals.

$$\int_C 3x^2 y \ dx \ + \ x^3 \ dy.$$

   (b) (5 points) Let $C$ be the curve in the plane with parametrization $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \leq t \leq 2\pi$. Calculate the following line integral using Green's Theorem.

$$\int_C 3xy \ dx \ + \ x^2 \ dy.$$

10. (10 points) Find the lateral surface area (ie area of the vertical side) of the semi-cylinder $x^2 + y^2 = 1$ with $x \geq 0$, bounded from above by the surface $z = xe^y$ and from below by $z = 0$.

11. (10 points) Use the change of variables $x = v$ and $y = u^2 + v$ to compute the integral

$$\iint_R \frac{e^{\sqrt{y-x}}}{x+2-y} dA$$

where $R$ is the triangle bounded by $x = y$, $x = 0$ and $y = 2$. (Hint: To find the limits of integration, determine what each of the bounding curves are in terms of $u$ and $v$.)

12. **Extra credit (5 points)**: Prove using chain rule that the gradient $\nabla f(x, y)$ of a function $f(x, y)$ is perpendicular to the level curve of $f$ that passes through $(x, y)$.