Math 255 Final Exam Dec. 15, 2015

Rules: You may use your textbook and your notes, but no other resources. You must work alone. Notice that each problem specifies which geometric axioms you may use: neutral, Euclidean, or hyperbolic. Each part of each problem is worth 10 points. Due Tues., Dec. 22 at noon.

1. (Neutral) Let $\square ABCD$ be a convex quadrilateral.
   (a) Show that if $\angle DAB \cong \angle CBA$ and $\overline{AD} \cong \overline{BC}$ then $\angle ADC \cong \angle BCD$.
   (b) Show that if $\angle DAB \cong \angle CBA$ and $\angle ADC \cong \angle BCD$ then $\overline{AD} \cong \overline{BC}$.
2. (Euclidean) Suppose that $\triangle ABC$ is a triangle, and A' and B' are points such that $A' * A * C$ and $B' * B * C$. Let $\overrightarrow{AD}$ be the bisector of $\angle A'AB$, and let $\overrightarrow{BE}$ be the bisector of $\angle B'BA$.
   (a) Show that $\overrightarrow{AD}$ and $\overrightarrow{BE}$ intersect. Let P be the point of intersection.
   (b) Show that P also lies on the bisector of $\angle ACB$.
   (c) Show that there is a circle centered at P that is tangent to all three of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.
3. (Neutral) Suppose that A and B are distinct points, and O is a point that does not lie on $\overleftrightarrow{AB}$.
   (a) Show that there is a unique line ℓ such that $O \in \ell$, $d(A, \ell) = d(B, \ell)$, and A and B are on opposite sides of ℓ. (Hint: Show that any such line ℓ must pass through the midpoint of $\overline{AB}$.)
   (b) Show that there is a unique line ℓ such that $O \in \ell$, $d(A, \ell) = d(B, \ell)$, and A and B are on the same side of ℓ. (Hint: Show that any such line must be perpendicular to the perpendicular bisector of $\overline{AB}$.)
4. (Hyperbolic) Let $\angle BAC$ be an angle, and let $\overrightarrow{AD}$ be the bisector of the angle.
   (a) Show that there is a line ℓ and points $P, Q, R \in \ell$ such that $P \in \overrightarrow{AD}$, $Q * P * R$, $\overrightarrow{PQ} \mid \overrightarrow{AB}$, and $\overrightarrow{PR} \mid \overrightarrow{AC}$. (The set $\ell \cup \overrightarrow{AB} \cup \overrightarrow{AC}$ might be described as a *doubly asymptotic triangle*.)
   (b) Show that the line ℓ is unique. In other words, show that if a line ℓ' and points P', Q', and R' are as described in part (a), then $\ell' = \ell$.

(continued on other side)

5. (Neutral) We often "drop a perpendicular" from a point to a line. You might wonder, "Can we 'drop' an angle other than 90°?" You'll show in this problem that you can.

   Suppose that ℓ is a line, P is a point not on ℓ, and $0 < \alpha < 90$. Show that there are distinct points Q and R on ℓ such that $\mu(\angle PQR) = \alpha$. (Hint: Let $\mu(\angle BAC) = \alpha$. Use a homework problem to show that there is a point E on $\overrightarrow{AB}$ such that $d(E, \overleftrightarrow{AC}) = d(P, \ell)$. Let F be the foot of the perpendicular from E to $\overleftrightarrow{AC}$. Use $\triangle AEF$ to help you locate the points Q and R on ℓ.)