Math 105, Fall 2016

Final Exam, Tuesday, December 20, 2016

Instructions: Do all ten numbered problems. If you wish, you may also attempt the three optional bonus questions. **Show all work**, including scratch work. Little or no credit may be awarded, **even when your answer is correct**, if you fail to follow instructions for a problem or fail to **justify your answer.** Unless otherwise noted, simplify your answers whenever possible. If you need more space, use the back of any page. If you have time, check your answers.

WRITE LEGIBLY. **NO CALCULATORS.**

1. **(40 points)** Evaluate each of the following limits. Be clear about whether the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. As always, justify your answers.

1a. $\displaystyle\lim_{x\to 5} \frac{x^2-2x-15}{x^2+2x-35}$

1b. $\displaystyle\lim_{x\to -2} \frac{x+2}{\sqrt{x+11}-3}$

1c. $\displaystyle\lim_{x\to 4} \frac{x^2+16}{x^2-16}$

1d. $\displaystyle\lim_{x\to 1} \frac{F(x^2)-3}{[F(x)]^2-9}$ where $F(x)=x+2$.

1e. $\displaystyle\lim_{x\to 1} \frac{x^2-7x+6}{(x-1)^3}$

1f. $\displaystyle\lim_{x\to 3^-} \frac{|x-3|}{x^2-2x-3}$

1g. $\displaystyle\lim_{x\to -\infty} \frac{x^2+3x^3-4x^4}{5x^4-14}$

1h. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x^2+5x}}{6x+7}$

2. **(20 points, 4 parts)** Compute the following derivatives. Simplify your answers.

2a. $f'(x)$, where $f(x)=\sqrt{\sqrt{x}+\dfrac{3}{\sqrt{x}}}$

2b. $g'(x)$, where $g(x)=\left(x^2-6\right)^4(x+4)^8$

2c. $\dfrac{dy}{dx}$, where $x^2+y^3=2xy^2-7$

2d. $K''(x)$, where $K(x)=\dfrac{x^2}{x^2+2}$. [Note: **second** derivative.]

3. **(15 points)** Let $F(x)=\dfrac{2}{x+7}$. Compute $F'(x)$ using the **limit definition of the derivative**.

4. **(10 points)** Consider the equation $y^2+xy-x^5=8-8x-x^2-y$. Find the **equation of the tangent line** to this curve at the point $(1,0)$.

5. **(15 points)** Find the absolute minimum and absolute maximum values of the function

$$g(x)=x^3-12x$$

on the interval $[-3,1]$.

6. **(20 points, 6 parts)** Consider the function $f(x)=\begin{cases}\sqrt{x-1} & \text{if } x>1\\ x^2-1 & \text{if } -2<x<1\\ \dfrac{1}{x+3} & \text{if } x\le -2\end{cases}$

6a. Carefully sketch the graph $y = f(x)$.

6b. State the **domain** of $f(x)$.

6c. Compute $\lim_{x \to 1} f(x)$.

6d. Is f continuous at $x = 1$? Why or why not?

6e. Compute $\lim_{x \to -2} f(x)$.

6f. Is f continuous at $x = -2$? Why or why not?

7. **(20 points)** A ladder that is 10 feet long is leaning against a vertical wall. The base of the ladder begins to slide away from the wall, and so the top slides down the wall. When the top of the ladder is 6 feet above the ground, it is sliding down the wall at $\frac{1}{2}$ foot per second. How fast is the base of the ladder sliding away from the wall at that moment?

8. **(25 points)** Let $f(x) = \dfrac{x(2x+5)}{(x+1)^2} = \dfrac{2x^2+5x}{x^2+2x+1}$. **Take my word for it that**:

$$f'(x) = \frac{-(x-5)}{(x+1)^3} \qquad \text{and} \qquad f''(x) = \frac{2(x-8)}{(x+1)^4}.$$

Analyze

- the domain of f,
- the vertical and horizontal asymptotes of f,
- intervals of increase or decrease of f,
- concavity of f, and
- x-coordinates of local extrema and inflection point of f.

Then use this information to present a detailed and labelled sketch of the curve $y = f(x)$.

[Note: It is OK to summarize your increasing/decreasing and concavity conclusions in charts instead of words. Also, you **do not** need to compute y-coordinates of maxima, minima, and inflection points.]

9. **(20 points)** A farmer needs to fence off a rectangular field and then divide the rectangle into three pens with two extra middles fence running parallel to two of the sides. The fencing for the outside fence costs \$3 per foot, while fencing for the middle fences costs \$2 per foot. What is the largest possible area the farmer can enclose in this way if he can only spend \$2400 on the fencing? [Warning: make sure you've set the total **cost** of the fencing equal to 2400, not the total length.]

10. **(15 points)** Find and classify (as local maximum, local minimum, or neither) all critical numbers of $h(x) = x^4(x-7)^3$.

OPTIONAL BONUS A. (2 points) We need a metal can in the shape of a cylinder with a volume of $1000\,\text{cm}^3$. What radius and height should the can have to minimize its surface area? (Remember that the surface area will include the top circle, bottom circle, and the cylinder part around the side.)

OPTIONAL BONUS B. (2 points) Compute $\lim_{x \to -\infty} \sqrt{x^4 - 5} - \sqrt{x^4 - 5x^2}$.

OPTIONAL BONUS C. (1 point) Name the current heads of state of India and North Korea. (For North Korea, I want the person's full name; for India, just the last name is fine.)