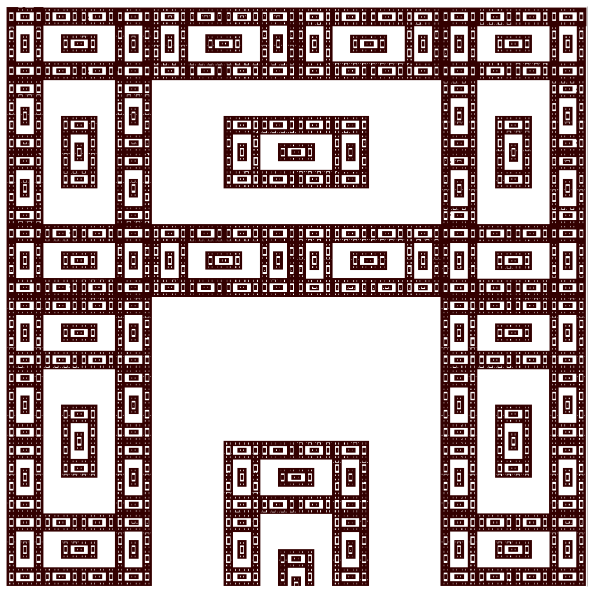
Math 225 • Fractal Geometry
Amherst College
Final Exam. Dec. 19, 2016

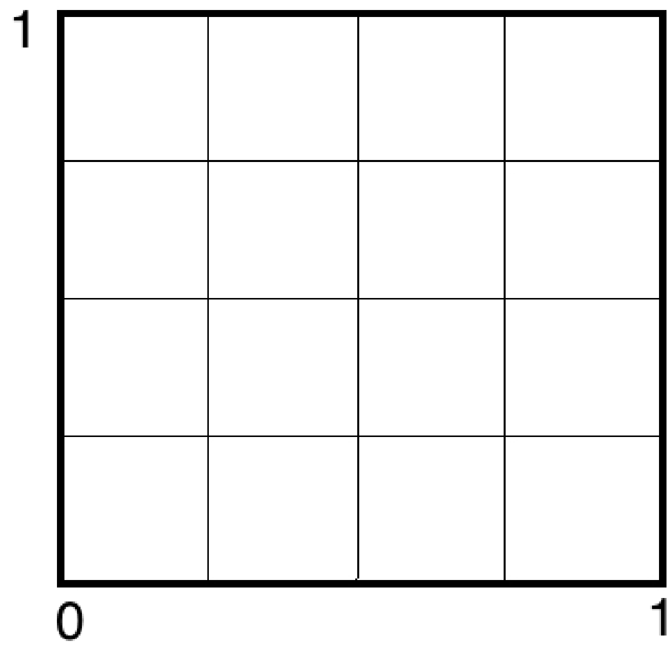
Problem 1(a). Find a set of IFS rules to generate the following attractor $\mathcal{B}$. As usual, $\mathcal{B}$ sits inside the unit square.
Note. You may not need to use all of the rows provided in the table.



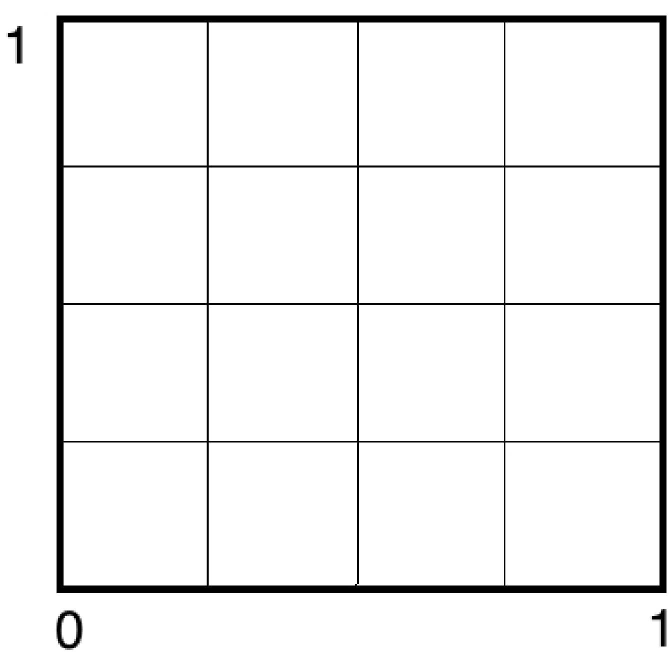
Please list your *angles in radians*

	r	s	θ	φ	e	f

Problem 1(b). Let S be the filled in unit square. Let $\mathcal{T}$ denote the collage map determined by your IFS from Problem 1(a). Sketch $\mathcal{T}(S)$ and $\mathcal{T}^2(S)$ in the grids provided:



T(S)



T^2(S)

Problem 1(c). Compute the Hausdorff distances $h(S, \mathcal{B})$, $h(\mathcal{T}(S), \mathcal{B})$, and more generally, $h(\mathcal{T}^n(S), \mathcal{B})$ for any $n \geq 0$.

Problem 1(d). Prove that the sequence $S, \mathcal{T}(S), \mathcal{T}^2(S), \mathcal{T}^3(S), \ldots$ converges to the fractal $\mathcal{B}$ in the Hausdorff metric.

Problem 2(a). `Figure 1`-`Figure 4` show 50,000 points obtained by randomly iterating the IFS $\{T_1, T_2, T_3\}$ for the gasket, with various probabilities attached to the contraction maps.

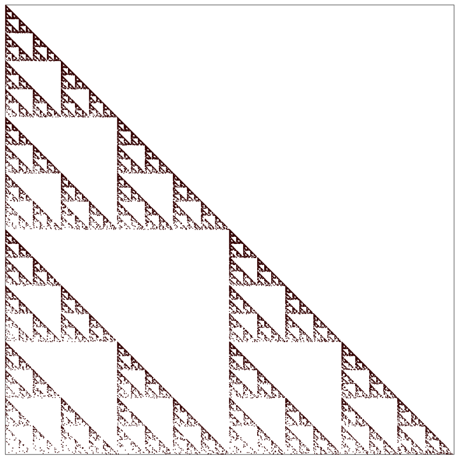


Figure 1

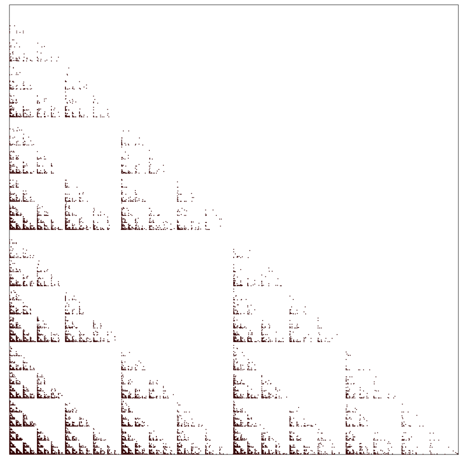


Figure 2

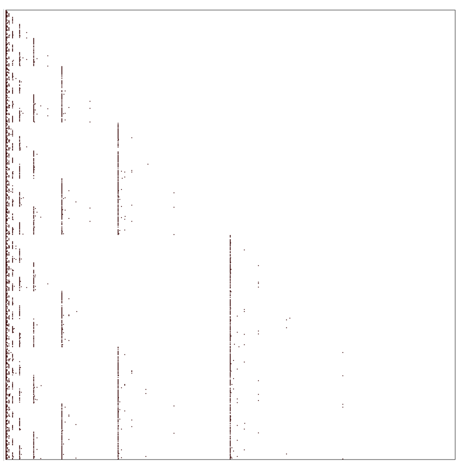


Figure 3

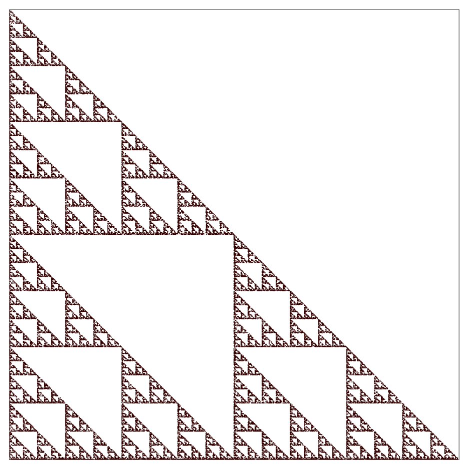


Figure 4

Match each figure to its corresponding probability set, each of which is one of the following:

Probability set #1: $\{\text{prob}(T_1) = \frac{1}{3}, \ \text{prob}(T_2) = \frac{1}{3}, \ \text{prob}(T_3) = \frac{1}{3}\}$

Probability set #2: $\{\text{prob}(T_1) = \frac{1}{5}, \ \text{prob}(T_2) = \frac{3}{10}, \ \text{prob}(T_3) = \frac{1}{2}\}$

Probability set #3: $\{\text{prob}(T_1) = \frac{1}{100}, \ \text{prob}(T_2) = \frac{1}{2}, \ \text{prob}(T_3) = \frac{49}{100}\}$

Probability set #4: $\{\text{prob}(T_1) = \frac{49}{100}, \ \text{prob}(T_2) = \frac{1}{100}, \ \text{prob}(T_3) = \frac{1}{2}\}$

Probability set #5: $\{\text{prob}(T_1) = \frac{1}{10}, \ \text{prob}(T_2) = \frac{1}{2}, \ \text{prob}(T_3) = \frac{1}{2}\}$

Probability set #6: $\{\text{prob}(T_1) = \frac{7}{10}, \ \text{prob}(T_2) = \frac{3}{20}, \ \text{prob}(T_3) = \frac{3}{20}\}$

Fill your answers in the following blanks (you will not need to use all six probability sets above, as there are only four figures):

Figure 1 = Probability set # ____________ Figure 2 = Probability set # ____________

Figure 3 = Probability set # ____________ Figure 4 = Probability set # ____________

Problem 2(b). Suppose we iterate the IFS for the gasket randomly with probability (as in part (a)). For this problem (Problem 2(b)), we seek to understand when we find the string 2223. There are five relevant states, A-E, which together, cover all possibilities. As usual, "-1" means "not 1", "-2" means "not 2", "-3" means "not 3":

$A :=$ left-most digit is -3, but left-most pair is not 23, left-most triple is not 223, and string 2223 has not occurred

$B :=$ left-most digit is 3, and string 2223 has not occurred

$C :=$???

$D :=$???

$E :=$???

States A and B are given. Determine states C, D and E.

Problem 2(c). Iterating randomly with probabilities as in Probability set #2 given in part (a), draw the corresponding transition probability graph arising from the states A-E from part (b).

Problem 2(d). Using the probabilities from part (c) and states from part (b), and supposing we begin in state A,

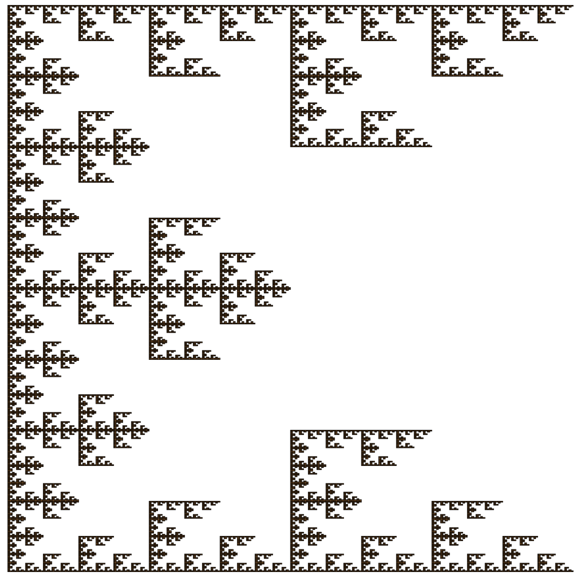
i) Compute prob(state C | after 3 iterates).

ii) Determine the smallest number $k \in \{1, 2, 3, 4, \dots\}$ for which

prob(state E | after k iterates) is positive.

iii) Using your k from part ii), compute the numerical value of prob(state E | after k iterates).

Problem 3(a). The fractal below is the attractor of a 1-IFS. What is the associated transition graph?

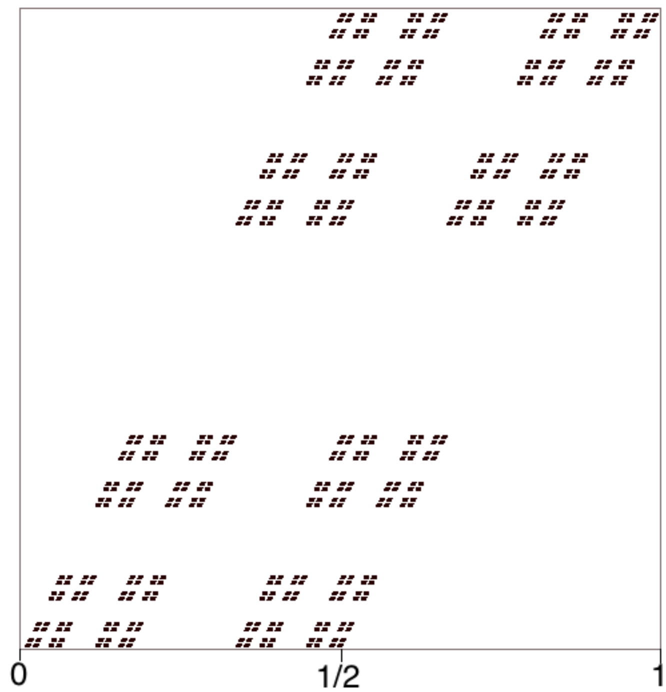


As usual, the fractal is enclosed by the unit square.

Problem 3(b). Can the fractal from part (a) be realized as the attractor of a 0-IFS (i.e., an IFS with no memory)? Justify your answer.

Problem 3(c). Determine the similarity dimension of the fractal from part (a). You may use any method you wish, but must fully and clearly justify your answer.

Problem 4(a). Find a set of IFS rules to generate the following fractal, F, below. For reference, the unit square is drawn in gray around the fractal F.
Note. You may not need to use all of the rows provided in the table.



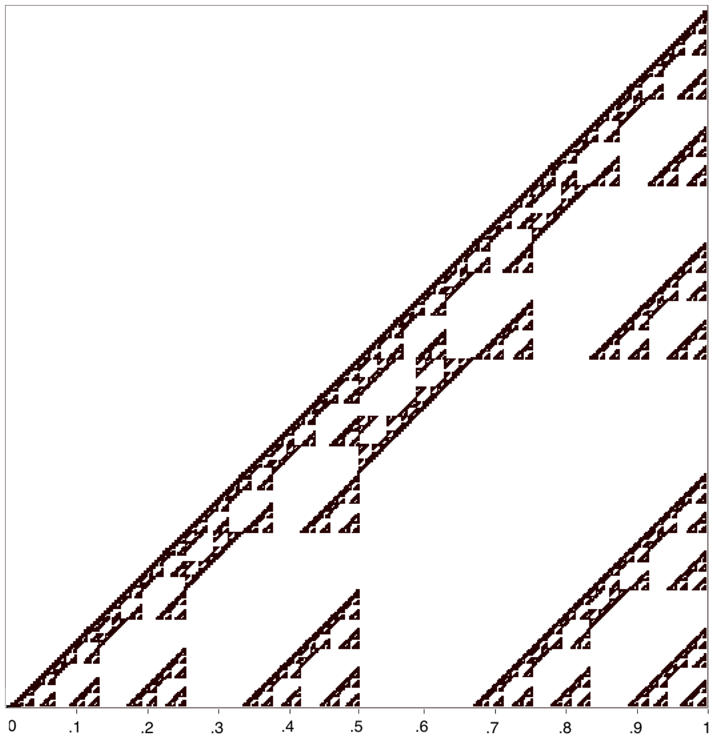
Please list your *angles in radians*

	r	s	θ	φ	e	f

Problem 4(b). What is the definition of $d_h(A)$, the Hausdorff dimension of a space A? Provide a formal and complete definition, clearly explaining and defining everything necessary to understand this definition.

Problem 4(c). Using your definition from part (b), prove that $\log(4)/\log(3)$ is an upper-bound for the Hausdorff dimension of the fractal F from part (a). *Do not use Hutchinson's Theorem.*

Problem 5(a). Find a set of IFS rules to generate the following fractal. For reference, the unit square is drawn in gray around the fractal.
Note. You may not need to use all of the rows provided in the table.



Please list your *angles in radians*

	r	s	θ	φ	e	f

Problem 5(b). Denote the fractal in Problem 5(a) by $\mathcal{A}$. Prove that its Hausdorff dimension satisfies

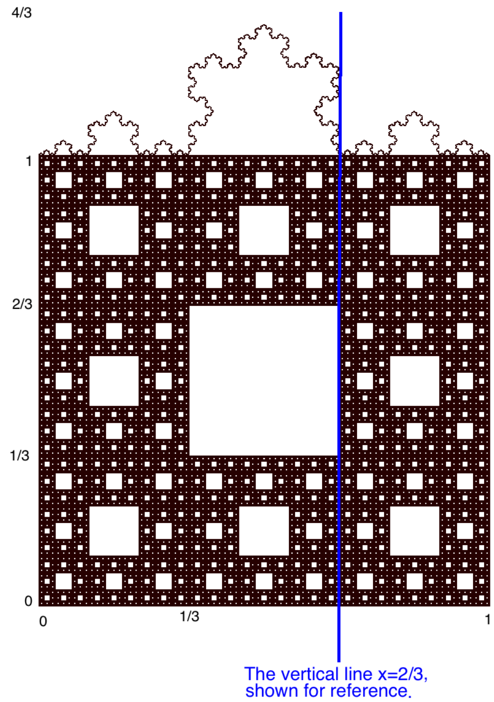
$$d_h(\mathcal{A}) \leq \frac{\log(3)}{\log(2)}.$$

You may use any method you wish, but must fully and clearly justify your answer.

Problem 6(a).

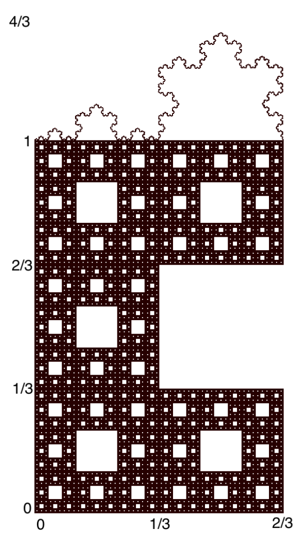
`Figure 1` below shows a Koch curve $\mathcal{K}$ sitting directly on top of the Sierpinski carpet $\mathscr{S}$. The Sierpinski carpet $\mathscr{S}$ sits inside the unit square, and the Koch curve $\mathcal{K}$ sits outside.

For reference, the line $x = 2/3$ is marked in blue, and some points along the axes are marked.

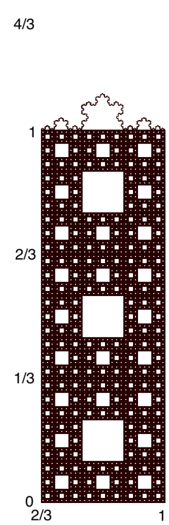


`Figure 1:` $\mathscr{S} \cup \mathcal{K}$.

We now split $\mathscr{S} \cup \mathcal{K}$ into two parts $\mathcal{F}_1$ and $\mathcal{F}_2$, shown in `Figure 2` and `Figure 3`:



`Figure 2:` $\mathcal{F}_1$



`Figure 3:` $\mathcal{F}_2$

Problem 6(a) continues on the next page.

Problem 6(a) (cont.) More precisely,

`Figure 2` depicts the set $\mathcal{F}_1$, defined to be the portion of $\mathscr{S} \cup \mathcal{K}$ with x-values $\leq 2/3$. That is, $\mathcal{F}_1 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathscr{S} \cup \mathcal{K} \mid x \leq \frac{2}{3} \right\}$.

`Figure 3` depicts the set $\mathcal{F}_2$, defined to be the portion of $\mathscr{S} \cup \mathcal{K}$ with x-values $\geq 2/3$. That is, $\mathcal{F}_2 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathscr{S} \cup \mathcal{K} \mid x \geq \frac{2}{3} \right\}$.

Compute $d_h(\mathcal{F}_1)$.
You may use any method you wish, but must fully and clearly justify your answer.

Problem 6(b). Using the definition of the box-counting dimension, determine $d_b(\mathcal{F}_3)$, the box-counting dimension of the set $\mathcal{F}_3$ shown below. As usual, this set $\mathcal{F}_3$ sits inside the unit square. *Do not use Hutchinson's theorem.*

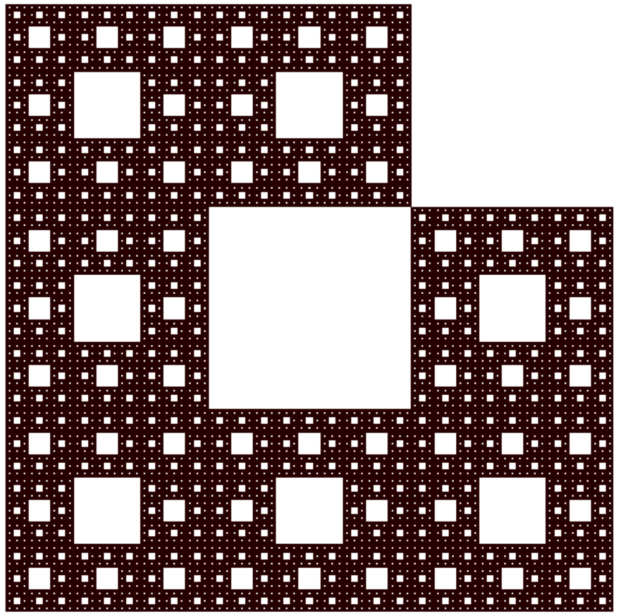


Figure 4: the set $\mathcal{F}_3$