Math 211: Multivariable Calculus, Fall 2017
Final Exam
Friday 22 December

1. Let $f(x,y) = \ln(2x+y)$.
   (a) (5 points) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(-1,3,0)$.
   (b) (3 points) Use a linear approximation to estimate the value $f(-1.1, 2.9)$.

2. Consider the function $f(x,y) = y\sin(\pi xy)$.
   (a) (3 points) In what direction does f increase most rapidly if you are at the point $(1,-1)$?
   (b) (2 points) What is the directional derivative of f at $(1,-1)$ in the direction you found in part (a)?
   (c) (3 points) In what directions does the directional derivative of f equal 0 if you are at the point $(1,-1)$?

3. (3 points) Suppose that the vector-valued function $\mathbf{r}(t)$ represents the position of a moving object, and suppose that the velocity vector $\mathbf{r}'(t)$ is perpendicular to the acceleration vector $\mathbf{r}''(t)$ for all values of t. Prove that the speed of the object is constant.

4. Consider the function

$$f(x,y) = \begin{cases} \dfrac{-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) (4 points) Compute the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.
   (b) (4 points) Is $f(x,y)$ continuous at $(0,0)$? Justify your answer.
   (c) (4 points) Is $f(x,y)$ differentiable at $(0,0)$? Justify your answer.

5. (5 points) A soccer ball is kicked at a speed of 10 meters per second, at an angle of $\pi/6$ radians to the horizontal. How far from the starting point does the ball hit the ground again? (Assume the ground is horizontal, there is no air resistance and the acceleration due to gravity is 10 m/s^2.)

6. (5 points) The graph of the function $\sin(2x)$ can be given parametrization

$$\mathbf{r}(t) = \langle t, \sin(2t) \rangle.$$

   Calculate the scalar curvature of this graph at the point $(\pi/4, 1)$.

7. (10 points) Find the absolute maximum and minimum values of the function

$$f(x, y) = x + 2y$$

subject to the constraint $x^2 + y^2 \leq 4$. Make sure you explain how you know your answers are the absolute maximum and minimum.

8. (8 points) Calculate the area of the part of the surface $z = x^2 + 3y$ for which (x, y) is contained in the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

9. (5 points) Use the change of variables $x = 2u$ and $y = 3v$ to calculate the the area of the region bounded by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

10. (5 points) Compute $\iiint_D z^2 \, dV$ where D is the region contained between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$.

11. (3 points each) Decide if each of the following vector fields $\mathbf{F}$ is conservative and, if so, find a function f such that $\mathbf{F} = \nabla f$. Give reasons for your answers.

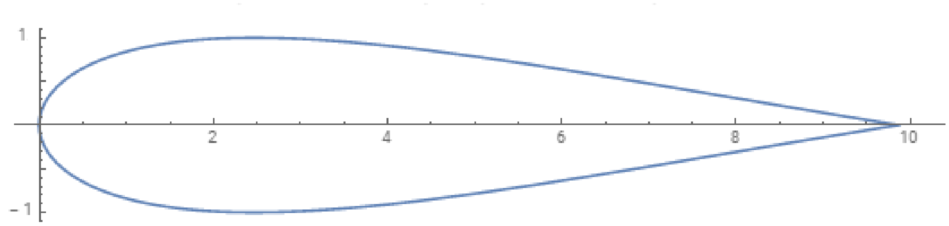
    (a) $\mathbf{F}(x, y) = \langle x^2 - \cos(2y), y^3 + 2x\sin(2y)\rangle$
    (b) $\mathbf{F}(x, y, z) = \langle 2xy - x^2, z^3, 3yz^2\rangle$

12. (4 points each) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in each of the following cases:

    (a) $\mathbf{F}(x, y, z) = \langle 2y, -3x, z\rangle$, C is the line segment going from $(1, 0, -1)$ to $(0, -2, 2)$
    (b) $\mathbf{F} = \nabla f$ where $f(x, y) = e^{x^2+y^2}$, C is part of the circle $x^2 + y^2 = 1$ with $x \geq 0$, oriented from top to bottom.
    (c) $\mathbf{F}(x, y) = \langle y^2, x^2\rangle$, C is the boundary of the square $[0, 1] \times [0, 1]$ oriented clockwise.

13. (8 points) Use Green's Theorem to calculate the area contained within the curve (pictured below) given by parametrization

$$\mathbf{r}(t) = \left\langle t^2, \sin(t)\right\rangle, \quad -\pi \leq t \leq \pi.$$



(You might need the formula $\int t\sin(t)\,dt = \sin(t) - t\cos(t) + c$.)

14. (5 points) Use Stokes' Theorem to calculate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \langle 0, 0, x \rangle$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ with $z \leq 2$, oriented with the upward-pointing normal vector.