Math 350: Groups, Rings and Fields
Section 02, Final Exam

- Attempt all problems, and **justify all your answers**.
- You may use theorems from class or the book to do so, but if it is not completely clear from context which theorem you are using, please briefly describe or state the theorem in question.
- Please write clearly and legibly.
- Please cross out any scratch work that you do not want graded.
- No notes, textbooks, calculators or outside help may be used on this exam.

1. (10 points) Define the following terms.
   (a) normal subgroup
   (b) index of a subgroup
   (c) field
   (d) maximal ideal
2. (10 points) Write down the orders of the following elements in the given group:
   (a) x^{21} in $\langle x\rangle$, where $|\langle x\rangle| = 70$.
   (b) $(3, 1)$ in $G = \mathbb{Z}_{12} \times \mathbb{Z}_6$
   (c) $\{2, 3, 4\}$ in $(\mathcal{P}(\{1, 2, 3, 4, 5\}), \Delta)$
   (d) $A_5(1\ 3)$ in S_5/A_5
3. (10 points) Let $\sigma = (1\quad 5)(2\quad 4\quad 3)$ and $\tau = (1\quad 4\quad 6)(3\quad 5\quad 2)$ be elements of S_7.
   (a) Express $\sigma\tau$ as a product of disjoint cycles, and use your answer to find the order of $\sigma\tau$.
   (b) Express $\sigma\tau$ as a product of transpositions and determine whether it is even or odd.
   (c) Is there an *even* permutation of order 6 in S_7? If so, give an example, and if not, explain why.
4. (10 points) Let G be group, let H be a subgroup of G, and let K be a normal subgroup of G. Define the subset J of G by
$$J = \{hk|h \in H, k \in K\}.$$
   Prove that J is a subgroup of G.
5. (10 points) Let R be a ring.

(a) Show that if I and J are ideals of R then $I + J = \{x + y \mid x \in I, y \in J\}$ is an ideal of R.

(b) Show that if I and J are maximal ideals of R and $I \neq J$, then $I + J = R$.

6. (10 points) Let $R = \mathbb{R}[X]$, and let $\phi : R \to \mathbb{R}$ be the function given by $\phi(f(X)) = f(1)$.

   (a) Show that ϕ is a ring homomorphism.

   (b) Show that ϕ is onto.

   (c) Since $\ker(\phi)$ is an ideal in R, it is principal. Find a generator for $\ker(\phi)$.

   (d) Is $\ker(\phi)$ a maximal ideal?

7. (10 points) Let $F = \mathbb{Z}_5$ and $R = F[X]$.

   (a) What are the zero divisors in R?

   (b) What are the units in R?

   (c) Is $f(X) = X^2 + X + 1$ irreducible in R?

   (d) Let I be the principal ideal generated by $f(X)$. Is R/I a field? What is the order of R/I?

   (e) The element $\overline{X}$ is a unit in R/I. What is its inverse?

8. (10 points) Let $(R, +, \cdot)$ be a nontrivial ring with unity such that for each nonzero $a \in R$, there exists a *unique* $b \in R$ such that $ab = 1$.

   (a) Show that R has no nonzero zero divisors.

   (b) Show that every nonzero element R is a unit.

9. (10 points) Let G be a group and let $\phi : G \to \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_6$ be a group homomorphism. Show that $a^{24} \in \ker(\phi)$ for all $a \in G$.

10. (10 points) Show that a group with at least 2 elements but no proper nontrivial subgroup must be finite and of prime order.