The derived moduli stack of logarithmic fat connections

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Based on the following papers:

- Lie groupoids and logarithmic connections, arXiv:2010.03685
- Normal forms and moduli stacks for logarithmic flat connections, arXiv:2209.00631
- The derived moduli stack of logarithmic flat connections, arXiv:2301.00962

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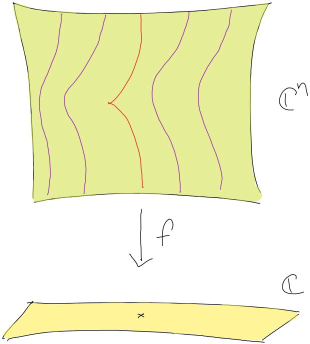
Logarithmic flat connections

We consider the following setting:

- $f : \mathbb{C}^k \to \mathbb{C}$ a weighted homogeneous function.
- Assume that $D = f^{-1}(0)$ is a Saito free divisor.



Goal: Classify flat connections on $\mathbb{C}^k$ with log singularities along D.

Example: Fuchsian ODEs

For $f = id : \mathbb{C} \to \mathbb{C}$, $D = \{0\}$. These are linear ODEs on $\mathbb{C}$ with Fuchsian singularity at the origin:

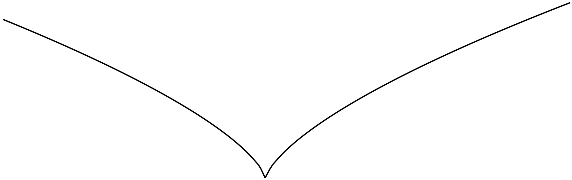
$$\frac{ds}{dz} = \frac{A(z)}{z} s,$$

where $A(z)$ is a holomorphic $n \times n$ matrix.

Studied by many people, such as Hukuhara, Turrittin, Levelt, Gantmacher, Babbitt and Varadarajan, Deligne, Simpson, Boalch, ...

Example: Cusp Singularity $x^2 = y^3$

For $f = x^2 - y^3 : \mathbb{C}^2 \to \mathbb{C}$, $D \subseteq \mathbb{C}^2$ is a curve with a cusp singularity at the origin:



Cusp Singularity $f = x^2 - y^3$

Logarithmic flat connection:

$$\nabla = d + \frac{A}{6f} df + \frac{B}{6f}(3xdy - 2ydx),$$

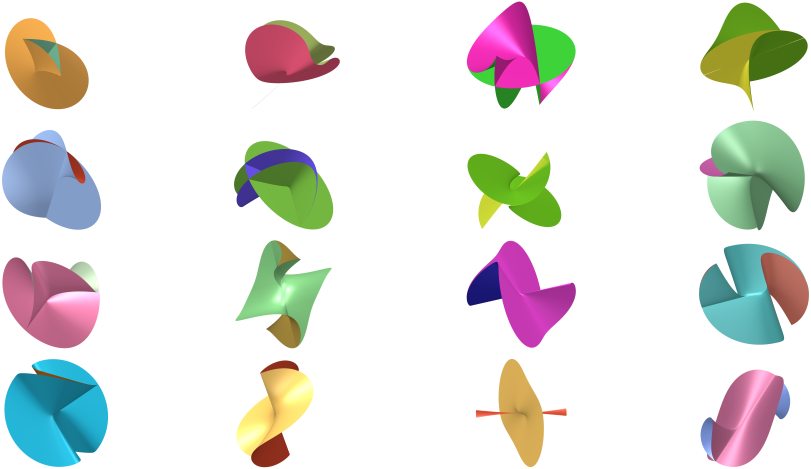
where $A(x, y), B(x, y) \in End(\mathbb{C}^n)$ holomorphic and satisfy

$$V(A) - E(B) + B + [A, B] = 0,$$

where

$$E = 3x\partial_x + 2y\partial_y, \qquad V = 3y^2\partial_x + 2x\partial_y.$$

There are many more examples, such as Sekiguchi's divisors in $\mathbb{C}^3$:



Definitions

Given $f : \mathbb{C}^k \to \mathbb{C}$ such that $D = f^{-1}(0)$ is a Saito free divisor.

- $T_{\mathbb{C}^k}(-\log D) = \{V \in T_{\mathbb{C}^k} \mid V(f) \in (f)\}$ defines a Lie algebroid.
- $\Omega^\bullet_{\mathbb{C}^k}(\log D) = \wedge^\bullet(T_{\mathbb{C}^k}(-\log D)^*)$, with de Rham differential d, defines a cdga.
- Given a Lie algebra $\mathfrak{g}$, ($\mathfrak{g} = Lie(G)$, G connected complex reductive) define

$$L_{D,\mathfrak{g}} = \Omega^\bullet_{\mathbb{C}^k}(\log D) \otimes \mathfrak{g}.$$

  This is a dgla with Lie bracket $[\alpha \otimes X, \beta \otimes Y] = \alpha \wedge \beta \otimes [X, Y]$.
- A logarithmic flat connection $\nabla = d + \omega$ is defined by $\omega \in L^1_{D,\mathfrak{g}}$ which satisfies the Maurer-Cartan equation:

$$F(\omega) = d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Moduli space of logarithmic flat connections

- The space of log flat connections is the Maurer-Cartan locus of $L_{D,\mathfrak{g}}$:

$$MC(L_{D,\mathfrak{g}}) = \{\omega \in L^1_{D,\mathfrak{g}} \mid F(\omega) = 0\}.$$

- $L^0_{D,\mathfrak{g}}$ is the Lie algebra of the gauge group $\mathfrak{G} = Map(\mathbb{C}^k, G)$, where $Lie(G) = \mathfrak{g}$. This acts on $MC(L_{D,\mathfrak{g}})$.
- The moduli space of flat log connections is the quotient stack

$$[MC(L_{D,\mathfrak{g}})/\mathfrak{G}].$$

- **Goal:** Find a finite-dimensional model for this stack.

Weighted homogeneous divisors

Now assume $f : \mathbb{C}^k \to \mathbb{C}$ weighted homogeneous. Namely $E(f) = rf$, for

$$E = \sum_i n_i z_i \partial_{z_i} \qquad (n_i > 0),$$

the infinitesimal generator of $\mathbb{C}^*$-action on $\mathbb{C}^k$.

$$\begin{tikzcd} \mathbb{C} \ltimes \mathbb{C}^k \arrow[r, shift left, "p"] \arrow[dr, "E"'] & \mathbb{C} \arrow[l, shift left, "j"] \\ & T_{\mathbb{C}^k}(-\log D) \arrow[u, "d\log f"'] \end{tikzcd} \qquad \begin{tikzcd} {[\mathcal{O}_{\mathbb{C}^k} \otimes \mathfrak{g}/\mathfrak{G}]} \arrow[r, shift right, "j^*"'] & {[\mathfrak{g}/G]} \arrow[l, shift right, "p^*"'] \arrow[d, "\pi^*"] \\ & {[MC(L_{D,\mathfrak{g}})/\mathfrak{G}]} \arrow[ul, "i^*"] \end{tikzcd}$$

$$\begin{array}{ccc} [\mathcal{O}_{\mathbb{C}^k} \otimes \mathfrak{g}/\mathfrak{G}] & \underset{j^*}{\overset{p^*}{\leftrightarrows}} & [\mathfrak{g}/G] \\ {\scriptstyle i^*} \nwarrow & & \swarrow {\scriptstyle \pi^*} \\ & [MC(L_{D,\mathfrak{g}})/\mathfrak{G}] & \end{array}$$

- Given $\omega \in MC(L_{D,\mathfrak{g}})$, its **residue** is $A = j^* i^*(\omega) \in \mathfrak{g}$. The **semisimple residue** S is the semisimple part of A.
- Define $MC(L_{D,\mathfrak{g}}, A)$ to be the space of log connections whose residue is in the adjoint orbit of A.
- Given $A \in \mathfrak{g}$ with semisimple component S, define

$$X_A := \{\omega \in MC(L_{D,\mathfrak{g}}, A), \mid i^*(\omega)_{ss} = p^* S\}, \qquad Aut(S) := Aut(p^* S).$$

- **Theorem (B.)** X_A is a finite-dimensional algebraic variety, $Aut(S)$ is an algebraic group acting on X_A, and there is an equivalence

$$[X_A/Aut(S)] \cong [MC(L_{D,\mathfrak{g}}, A)/\mathfrak{G}].$$

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Bundles of curved differential graded Lie algebras

Definition (Behrend, Ciocan-Fontanine, Hwang, and Rose, 2014)

A *bundle of curved differential graded Lie algebras* over a manifold M consists of a graded vector bundle $\mathcal{L}^\bullet$ starting in degree 2, which is equipped with the following data

1. a section $F \in \Gamma(M, \mathcal{L}^2)$,
2. a degree 1 bundle map $\delta : \mathcal{L}^\bullet \to \mathcal{L}^\bullet[1]$
3. a smoothly varying graded Lie bracket $[-,-]$ on the fibres of $\mathcal{L}^\bullet$,

satisfying the following conditions

1. the Bianchi identity $\delta F = 0$,
2. $\delta^2 = [F, -]$,
3. δ is a graded derivation of the bracket $[-,-]$.

A group G acting equivariantly on $(M, \mathcal{L}, F, \delta, [-,-])$ defines a *derived stack*.

Bundles of curved differential graded Lie algebras

Let $\mathcal{M} := G \curvearrowright (M, \mathcal{L}, F, \delta, [-,-])$ be an equivariant bundle of curved dgla's.

- The **Maurer-Cartan locus** is defined as the vanishing locus of F. It is preserved by the G-action, giving rise to a groupoid:

$$MC(\mathcal{M}) = G \ltimes V(F).$$

- Given a point $x \in V(F)$, the **tangent complex** is given by

$$\mathbb{T}_x\mathcal{M} = \mathfrak{g} \xrightarrow{\rho} T_xM \xrightarrow{dF_x} \mathcal{L}^2|_x \xrightarrow{\delta_x} \mathcal{L}^3|_x \to \dots$$

The derived stack of logarithmic flat connections

Given the dgla $L_{D,\mathfrak{g}}$, define the following bundle of curved dgla's $\mathcal{M}_{D,\mathfrak{g}}$:

- The base manifold is $M = L^1_{D,\mathfrak{g}}$,
- The bundle of graded Lie algebras is $\mathcal{L} = M \times L^{\geq 2}_{D,\mathfrak{g}}$,
- The section is the curvature map $F(\omega) = d\omega + \frac{1}{2}[\omega, \omega]$.
- The differential is $\delta_\omega = d + [\omega, -]$.

The gauge group $\mathfrak{G}$ acts on $\mathcal{M}_{D,\mathfrak{g}}$ and defines the derived stack of logarithmic flat connections

$$[\mathcal{M}_{D,\mathfrak{g}}/\mathfrak{G}].$$

Given $A \in \mathfrak{g}$, let $\mathcal{M}_{D,\mathfrak{g}}(A)$ be the derived manifold whose base consists of $\omega \in L^1_{D,\mathfrak{g}}$ such that $j^* i^*(\omega)$ is in the adjoint orbit of A. Then

$$[\mathcal{M}_{D,\mathfrak{g}}(A)/\mathfrak{G}]$$

is the derived stack of log flat connections with residue conjugate to A.
Goal: Find a finite-dimensional model.

Finite-dimensional model

Let $\alpha_0 = \frac{1}{r} d\log(f)$, let $E \in T_{\mathbb{C}^k}(-\log D)$ be the weighted Euler vector field, and let $S \in \mathfrak{g}$ be semisimple. Define operators on $L_{D,\mathfrak{g}}$:

- $\delta_S = d + \alpha_0 ad_S$ is a degree $+1$ derivation such that $\delta_S^2 = 0$.
- ι_E is a degree -1 derivation.
- $L_S := L_E + ad_S$ is a degree 0 derivation.

Then

- $L_{D,\mathfrak{g},0} = \ker(L_S)$ is a finite-dimensional subalgebra which is preserved by δ_S. $\implies$ finite dimensional dgla $(L_{D,\mathfrak{g},0}, \delta_S)$.
- $L^0_{D,\mathfrak{g},0} \cong Lie(Aut(S))$.
- $U_0 = \ker(\iota_E) \cap L_{D,\mathfrak{g},0}$ is a sub-dgla and

$$(L_{D,\mathfrak{g},0}, \delta_S) \cong T[-1](U_0, \delta_S).$$

Finite-dimensional model

- Both $(L_{D,\mathfrak{g},0}, \delta_S)$ and (U_0, δ_S) define derived stacks $[\mathcal{W}_S/Aut(S)]$ and $[\mathcal{U}_S/Aut(S)]$ and

$$[\mathcal{W}_S/Aut(S)] \cong T[-1][\mathcal{U}_S/Aut(S)].$$

- Given $A \in \mathfrak{g}$, we again define a substack

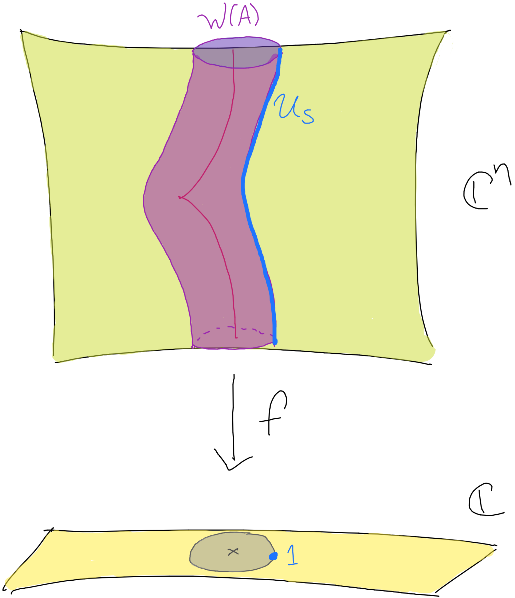
$$[\mathcal{W}(A)/Aut(S)] \subset [\mathcal{W}_S/Aut(S)]$$

of connections with residue in the adjoint orbit of A.

- Intuition: $[\mathcal{U}_S/Aut(S)]$ is a moduli space of flat connections on the fibre $f^{-1}(1)$ and the map

$$[\mathcal{W}(A)/Aut(S)] \to [\mathcal{U}_S/Aut(S)]$$

corresponds to ‘projecting out’ the unipotent part of the monodromy.


W(A)
$\mathcal{U}_s$
$\mathbb{C}^n$
f
$\mathbb{C}$
1

Main theorem

Theorem (B.)

The Maurer-Cartan locus of $\mathcal{W}(A)$ is isomorphic to X_A. Furthermore, there is an equivalence of derived stacks

$$q : [\mathcal{W}(A)/Aut(S)] \to [\mathcal{M}_{D,\mathfrak{g}}(A)/\mathfrak{G}].$$

By equivalence we mean that

1. q induces an equivalence of categories

$$Aut(S) \ltimes MC(\mathcal{W}(A)) \to \mathfrak{G} \ltimes MC(\mathcal{M}_{D,\mathfrak{g}}(A)).$$

2. For every point $\omega \in MC(\mathcal{W}(A))$, q induces a quasi-isomorphism

$$dq_\omega : \mathbb{T}_\omega \mathcal{W}(A) \to \mathbb{T}_{q(\omega)} \mathcal{M}_{D,\mathfrak{g}}(A).$$

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Large enough S

Theorem (B.)

If S is "large enough", then $[\mathcal{W}(A)/Aut(S)] \cong [dV(B)/P_S]$.

- $P_S \subseteq C_G(e^{-\frac{2\pi i}{pq}S})$ is a parabolic subgroup associated to the real part of S.
- $dV(B)$ is the derived vanishing locus of a function $B : Lie(P_S) \times H^1(U_0) \to H^1(U_0)$.

Example

Let $G = GL(n)$ and let

$$A = pq \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_n \end{pmatrix},$$

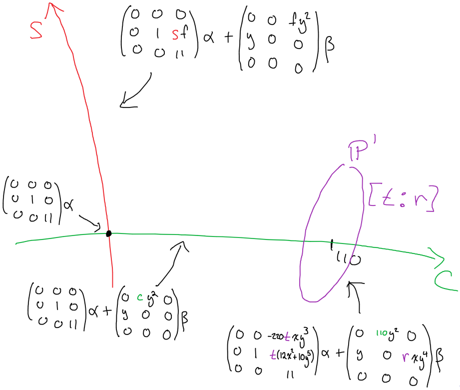
where m_i are distinct integers. Then

$$[\mathcal{W}(A)/Aut(S)] \cong [T^*FL_n/GL(n)].$$

S not large enough

If S is not large enough, then the theorem can fail. Take $f = x^2 - y^5$,

$$G = GL(3) \text{ and } S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{pmatrix}.$$



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The case $f = id : \mathbb{C} \to \mathbb{C}$

- Boalch studied the Riemann-Hilbert problem for flat G-connections on the disc $\mathbb{D}$ with logarithmic singularities at the origin (i.e. the case of Fuchsian ODEs).
- Let $\mathrm{Log}_{fr}(A)$ be the moduli space of **framed** log connections on the disc with residue in the adjoint orbit of A.
- Let $\mathrm{Loc}_{fr}(S^1)$ be the moduli space of framed G-local systems on S^1. By evaluating the monodromy, we get an isomorphism

$$\mathrm{Loc}_{fr}(S^1) \cong G.$$

- Boalch showed that $\mathrm{Log}_{fr}(A)$ is a quasi-Hamiltonian G-space (in the sense of AMM) with group valued moment map given by computing monodromy around the boundary:

$$\Phi : \mathrm{Log}_{fr}(A) \to G.$$

Shifted symplectic interpretation

- The moduli space of G-local systems on the circle has a $+1$-shifted symplectic structure (in the sense of PTVV)

$$\mathrm{Loc}(S^1) = \mathrm{Loc}_{fr}(S^1)/G \cong [G/G].$$

- Safronov: The group valued moment map descends to a Lagrangian map

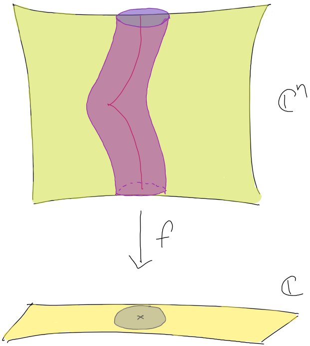
$$\Phi : \mathrm{Log}(A) = [\mathrm{Log}_{fr}(A)/G] \to [G/G].$$

- In particular, $\mathrm{Log}(A)$ has a Poisson structure.

Higher dimensional generalization

Consider again the setting $f : \mathbb{C}^k \to \mathbb{C}$.

- $X = f^{-1}(\mathbb{D})$ is a tubular neighbourhood of D.
- It's boundary $\partial X = f^{-1}(S^1)$ is a manifold of dimension $2k - 1$.
- $\mathrm{Loc}(\partial X)$ will have a $(3 - 2k)$-shifted Poisson structure.



Restriction to ∂X and taking monodromy defines a map

$$\Phi : [\mathcal{W}(A)/Aut(S)] \to \mathrm{Loc}(\partial X).$$

Higher dimensional generalization

Restriction to ∂X and taking monodromy defines a map

$$\Phi : [\mathcal{W}(A)/Aut(S)] \to \mathrm{Loc}(\partial X).$$

Conjecture

The moduli space $[\mathcal{W}(A)/Aut(S)]$ admits a $(2-2k)$-shifted Poisson structure and the map Φ admits a shifted coisotropic structure.

Shifted Poisson geometry for $x^p = y^q$

Let $f = x^p - y^q$, fix a semisimple matrix S, and consider the connection

$$\nabla = d + Sd \log f.$$

Theorem (B)

A formal neighbourhood of $\nabla \in [\mathcal{W}(S)/Aut(S)]$ admits a -2-shifted Poisson structure.

Thank You