① Hannah Burnam

Pulsar I

An Introduction to Topology

Mr. Zhang

Topology → study of shapes + spaces (2-D + 3-D) — qualitative properties

all are closed loops

no calculations

#2 studies those properties $\boxed{\text{invariant}}$ under deformation

VS.

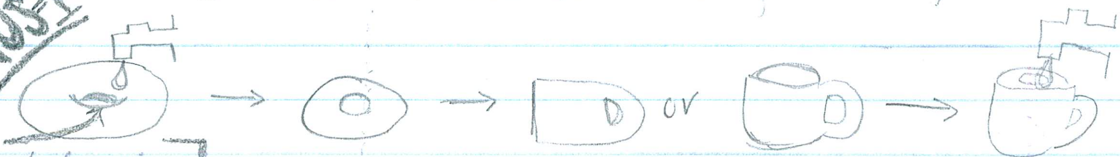
geometry: quantitative measurements (ex.)

Wikipedia → for things you don't know

Topologist ① one who studies topology

② someone who holds coffee using a doughnut in bfast

GENUS=1



this hole is invariant

very good picture!

(our minds can also enlarge shapes)

Steps

stretch
compress
bend
twist

} deform (good) after this if properties don't change then it is topological

tear
glue

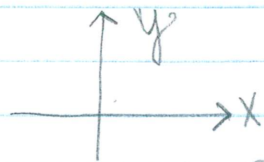
} X (bad)

What is the shape of our universe?

What are all the possible shapes of our universe?

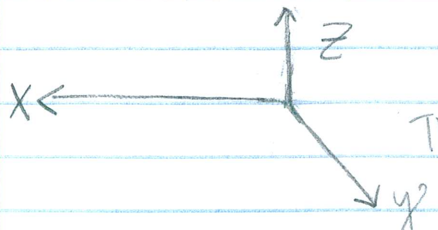
demension?

good summary ★ where you find elements: chemistry, sculpture, music, fashion design



(x, y) ordered pairs
$\boxed{2}$ numbers

The blackboard is 2-D



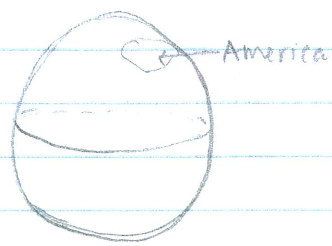
The classroom is 3-D

<u>locally</u>

the universe is <u>locally</u> 3-D
↳ at any point in time our neighbo[illegible]
space is like a classroom

<u>globally</u>

What about the global pic?



<u>**BIG QUESTION**</u>

assumptions: #$\boxed{1}$ no boundary/end (keep walking?)
#$\boxed{2}$ connected - (go from point A to point B)
- (disconnected is where no space exists between diff points)

#$\boxed{3}$ at any point in the universe there is a neighbor space which looks like a classroom (3-D) LOCALLY

BC we are 3D also this is hard for us to depict topology, it would be easier if we understood more dimensions i.e. 4-D

2

Honman

Principle: if something is too hard, then try a diff problem #1 related to the first one but #2 a lot simpler

EASIER Q: What are all the possible shapes of a 2-D universe?

#1 no boundary

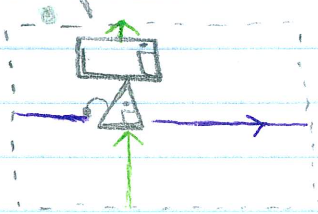
#2 connected

#3 locally (which is a blackboard)

* *Flatland* by Edwin A Abott in 1884*

citizens are shapes △ □ ○

shape of universe locally is a blackboard



rectangle north to south, return in 2 week

very good color combination

walks east w/ purple thread trail

after 2 months he comes back from the west to his start

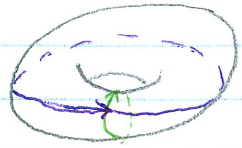
but a ~~sphere~~ is only the surface of a ball

w/ a thickness only of a point

- somewhere along the way there's an intersection, but the green team doesn't see the real thread ever (disproves THEORY I)

THEORY I $\boxed{1}$

THEORY II $\boxed{1}$

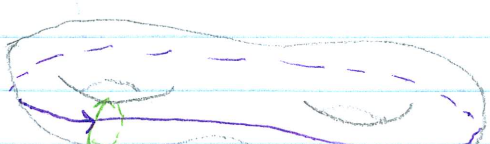


surface of a doughnut : torus

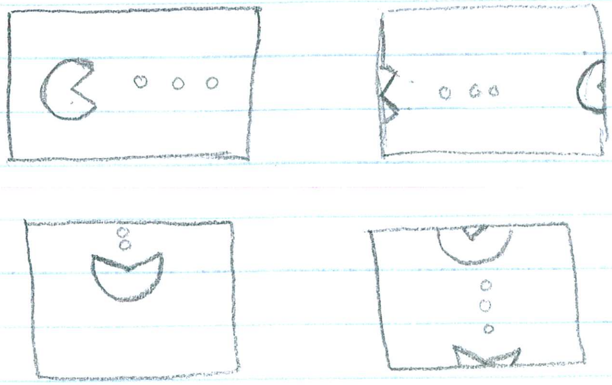
fits all properties

S^2 2-D sphere T^2 2-D torus

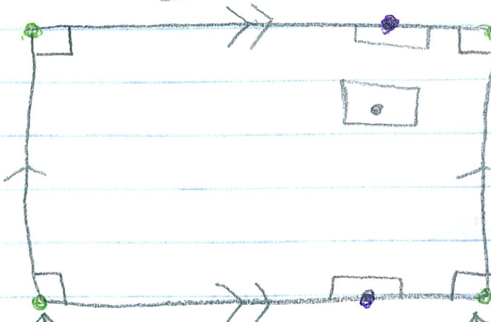
Theory III



torus w/ 2 holes $2T^2$



[EX1]



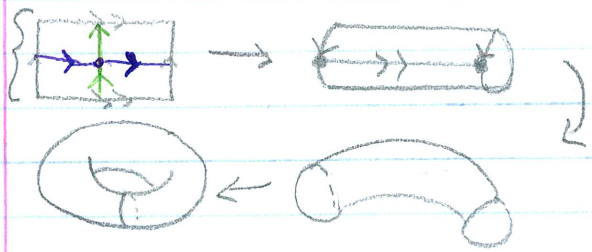
the same, glued abstractly

way >>/> face is how they attach

What is this space?

2-D

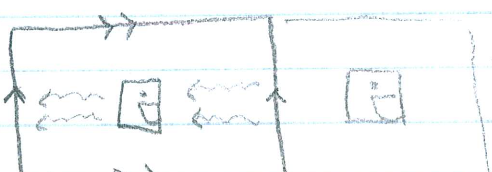
no 2nd intersection



#[1] no boundary
#[2] connected
#[3] locally a blackboard

are all same point

tic-tac-toe ← finite
torus game app by Jeff Weeks
↓
free lance mathematician



sees back of self
bc of light reflection

if universe repeats like this then it's most likely finite

A good philosophical question to reflect on → [if there are ∞ many, then which is the original

"Homework": Can you construct a 3-D universe using patterns similar to the one we did in [EX1],
remember conditions

③

Hannah

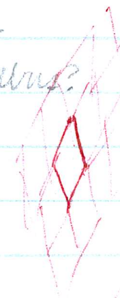
subject : fashion design

$\boxed{\text{or}}$

music

I will keep these two subjects in mind.

"HW" : equilateral shapes
parallelogram, rhombus?



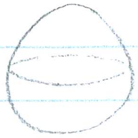
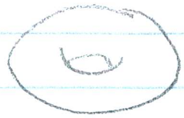
really good start. They give alternati models for T^2.

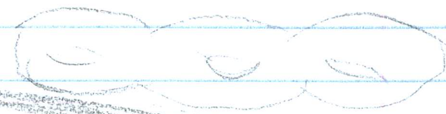
As we saw in class, to get 3D shapes, we started from a cube (which is 3D) and then glued opposite faces to get T^3.

Even though it is not for 3D, it opens ways for other interesting results.

4

Hannah

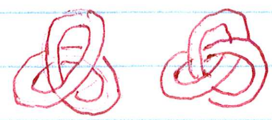
 S^2  T^2  $2T^2$

 $3T^2$

what is the diff between a square + a rectangle in terms of torus?

isotopy

homotopy $\simeq$

 T^2 

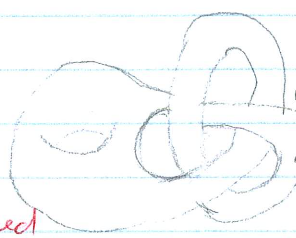
All T^2, A, B are homeomorphic

↓

same space

But A + B aren't isotopic

one cannot be deformed to another in 3-D

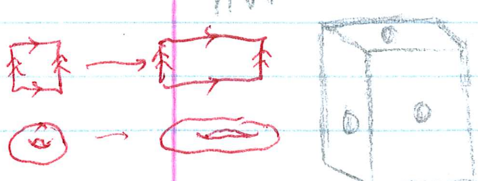
 $2T^2$

isotopic? yes

 $2T^2$

There are no difference topologically because one can be deformed to the other.

"HW"

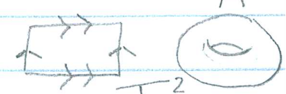


Outstanding

I like your way of organizing notes. I learned a lot from it !!

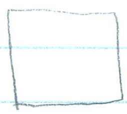
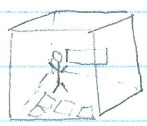
⑤
Hannah
Burman
PI

3-D Universe
1. no boundary
2. connected
3. locally a blackboard

using a procedure similar to  T^2 A

A + B are homeomorphic (same)
But not isotopic in 3-D  B
isotopic: can be deformed to the other

deformation: bend, twist, stretch compress ✓
tearing, gluing the incorrect parts X

 2-D blackboard  3-D classroom

1. glue the top face to the bottom face


cool
T^3

2. glue the left face to the right face
3. glue the front face to the back face

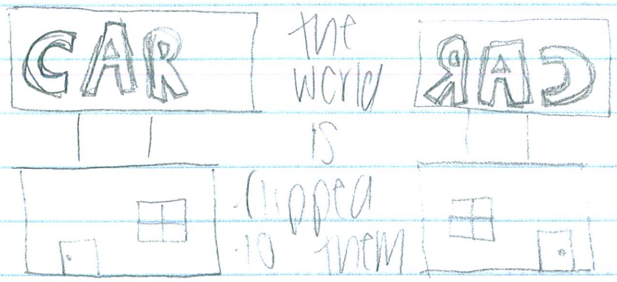
it's all about perspective: walking in a straight line
or reappearing out of walls

∞ copies in all directions

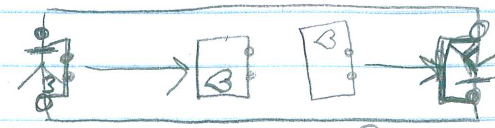
Another Question: Can our Universe be like this? A T^3?

DAY 3

flatlander exploration



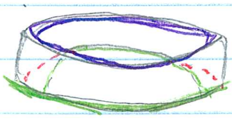
the world is flipped to them

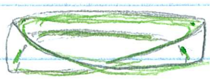


This is just one element in space

switches sides like a flexed mirror

the universe is not orientable

"U"  usual band

"M"  band (Möbius)

DIFFERENCES

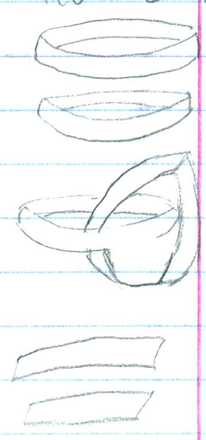
#① Möbius band has a 180° twist whereas usual has no twist

#② U has 2 sides but M has only 1

#③ Cut our M band along its midline / center line. What do we get?

- U band is just 2 U bands
- M band becomes an extra long and very twisted; a new band

Theories #③



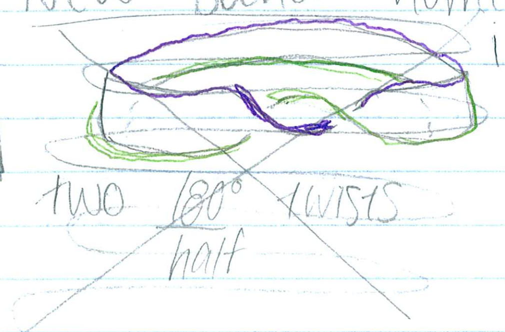
I

II

III

2 M bands IV

New Band: homeomorphic to U band + is not isotopic



~~two 180° half twists~~

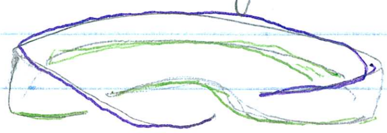
4 half (180°) twists

(6)
Hannah

TASK 2: Step #1 make Möbius band
Step #2 cut it along the 1/3 line

1: band w/ 4(1/2-twis
two linked bands
1: Möbius

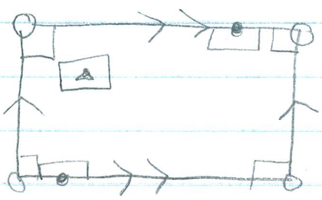
TASK 3: #1 construct a band with 2 half-twists
#2 cut along the midline



→ 2 linked bands ea. w/ 2(1/2-twists)

TASK 4: #1 construct a band w/ 3 1/2 twists
#2 cut it along midline
result: *trefoil knot*, found in chemistry
[twist to remove chemical bonds]

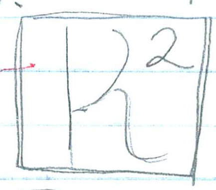
BACK TO THE STORY!

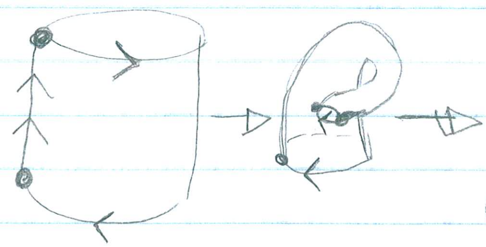


follows 3 rules!

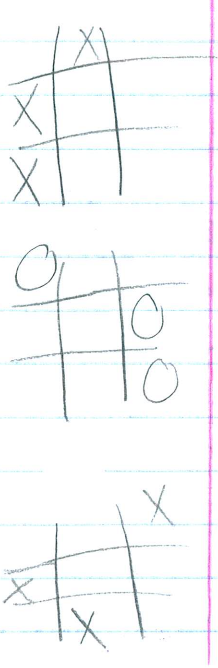
Klein <<<<<<<<
>>>>> bottle
↓
German mathematician

cool →

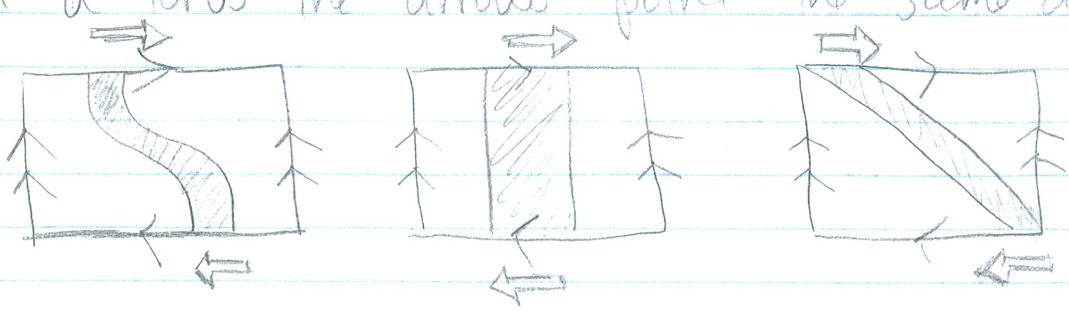




There's only 1 side like an M band

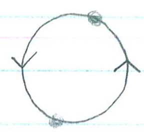


In a torus the arrows point the same direction



Matlab
Mathematica
maple
} math programs

PROJECTIVE PLANE

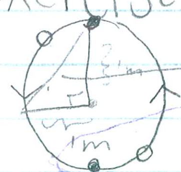

$\to P^2$

compare


S^2

Boy's surface
↳ Hilbert's student

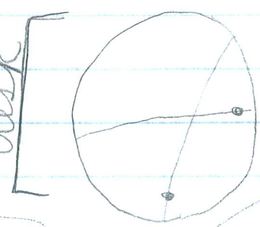
Exercise:

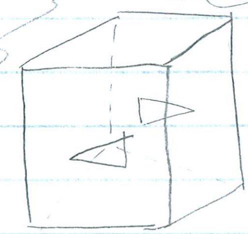


$\to \sqrt{2}\,m \approx 1.414\ldots m$

radius = 1m
What is the longest distance in this space?

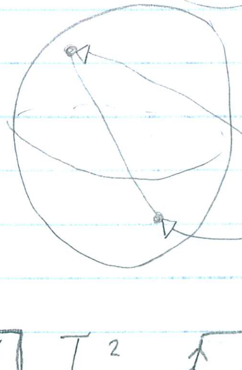
desk





$S^1 \times K^2$

glue top to bottom, left to right as usual, glue front to the back with left-right flip



solid shape

Projective space P^3
(same points)

can't	#1	T^2	torus: surface of a doughnut
contain M	#2	S^2	sphere: surface of a basketball
can	#3	P^2	projective plane: (boy's surface)
contain M	#4	K^2	Klein bottle:
[illegible] ☆	#5	M	Möbius band: not orientable

[[one can avoid being reflected]]

(if you walk in this space after you come back you will be mirror → reflected) [Go Around Twice]

7
Hannah

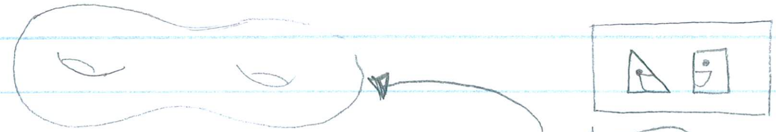
≈ means similar to

means my questions or wonders that I'm thinking about

get it!

good question.

Connected sum has what to do with snaps in music?

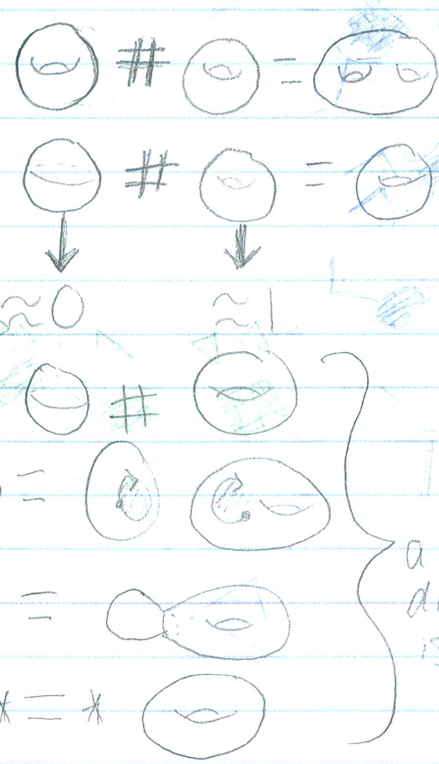


How do we explain this shape to a flatlander?

Start with: KEY IDEA → # ≈ +

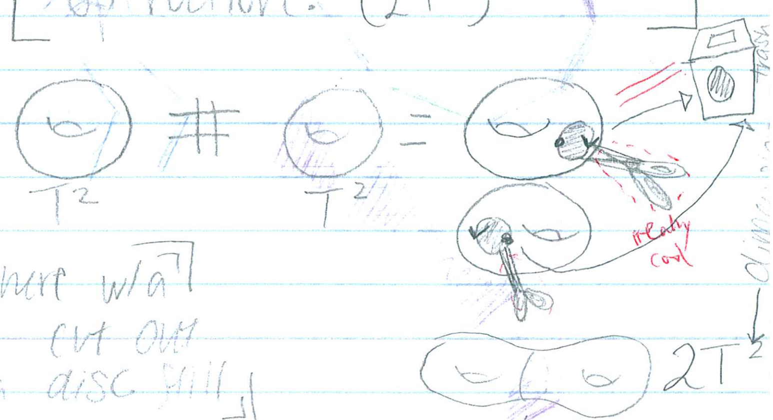
CONNECTED SUM : SUM : $1+1=2$ $0+2=2$ $0+1=1$ $0+n=n$

T^2 is formed by



≈0 ≈1

So, then what is this operation? ($2T^2$)



T^2 T^2

really cool

dimension

$2T^2$

(two holes)

so these are deformed (w/ the wows)

a sphere w/ a disc cut out is a disc still

Summary/observation : # w/ any surface
- cut off a disc then glue it back
- did nothing
- same surface

homeomorphic?

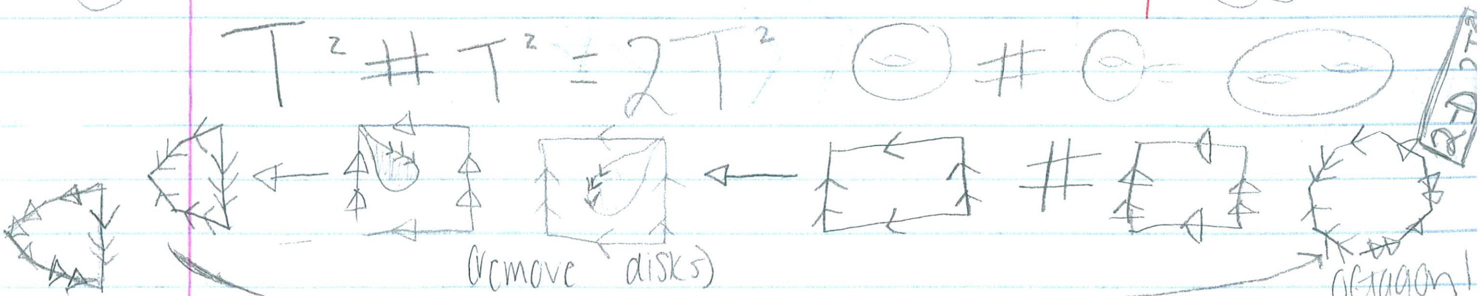
yes!

$K^2 + S^2 = K^2$

$T^2 + S^2 = T^2$

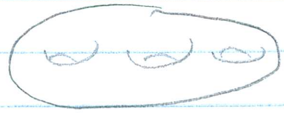
disks:

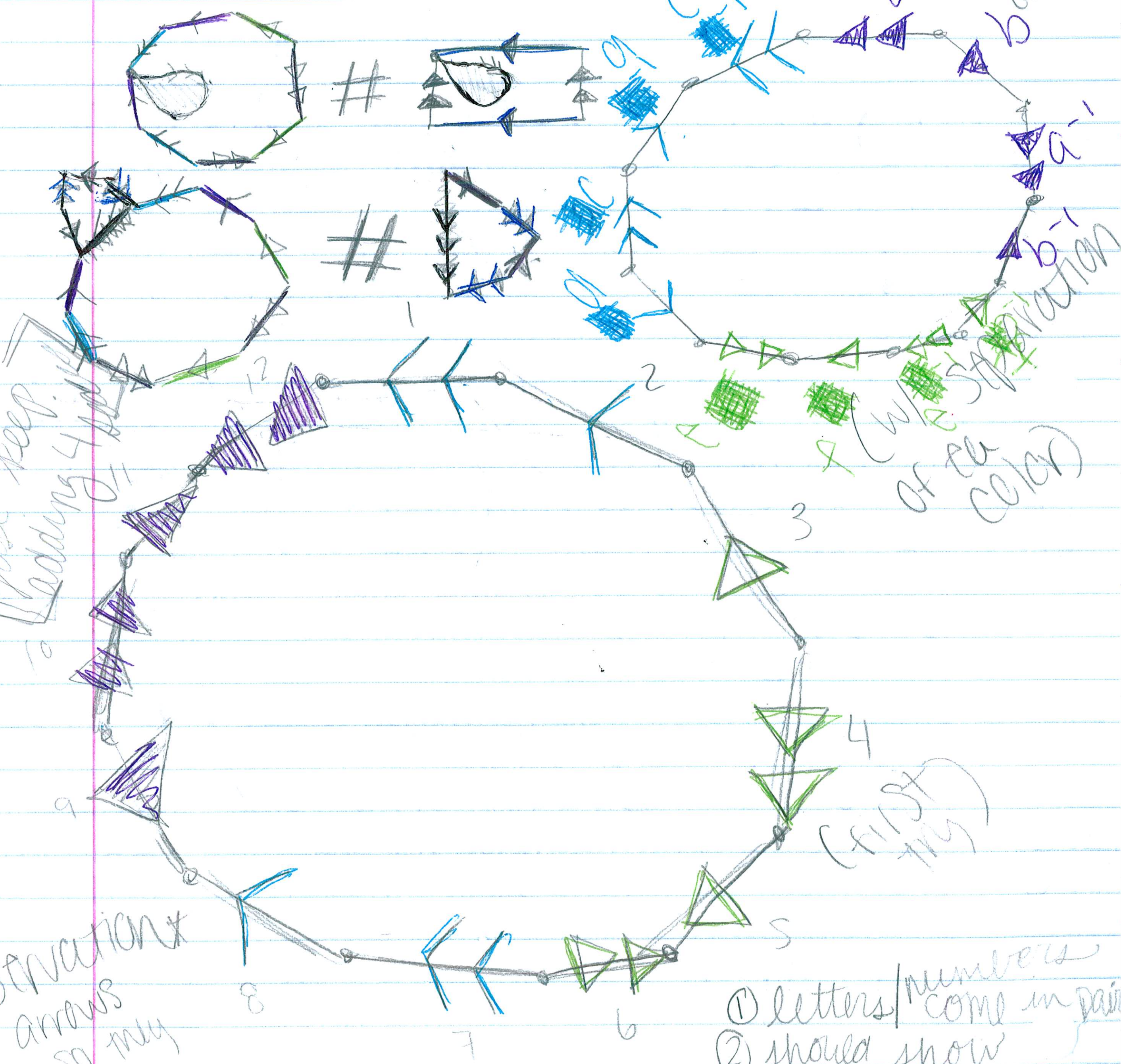
$T^2 \# T^2 = 2T^2$



(remove disks)

octagon!

What is a 2-D model of  ?
And how can we explain it to a flat lander?



Observation the arrows point so they can meet in the middle i.e. >—<

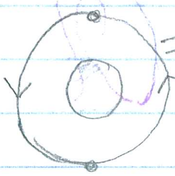
① letters/numbers come in pairs
② should show up alternating
③ i.e. $a + a^{-1}$

(8)

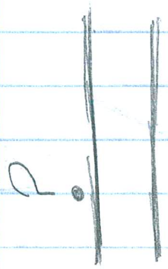
Hannah

$P^2 \setminus D^2$

projective plane with a disc cut off : \ = cut off

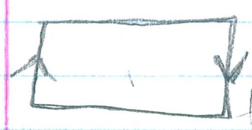


#① Can this be a disc? Nope
A disk contains Möbius strip (M)
But $P^2 \setminus D^2$ does



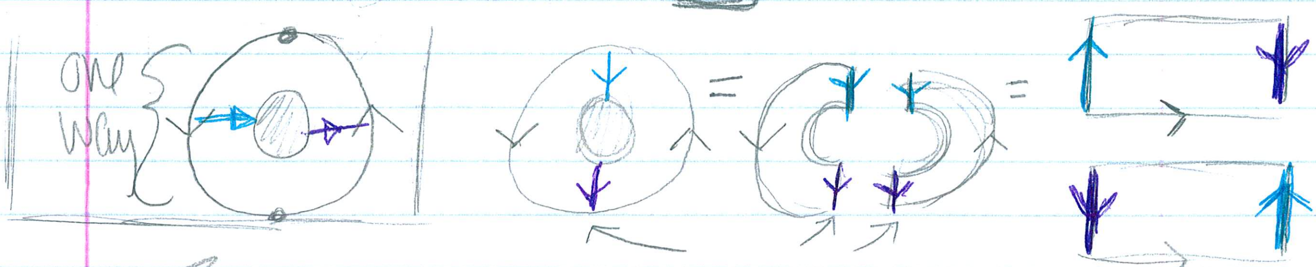
#② Can this be a torus?
No, for the same reason

#③ Can this be a Klein bottle?
(no boundary) 2-D
but cutting out a disc leaves a circle containing a boundary

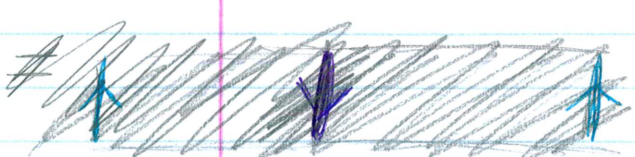


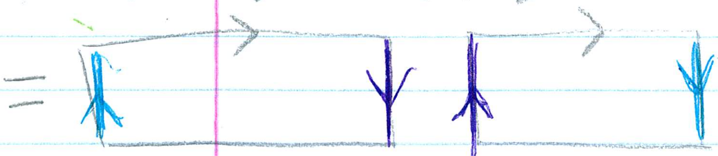
Also No bc $P^2 \setminus D^2$ has a boundary

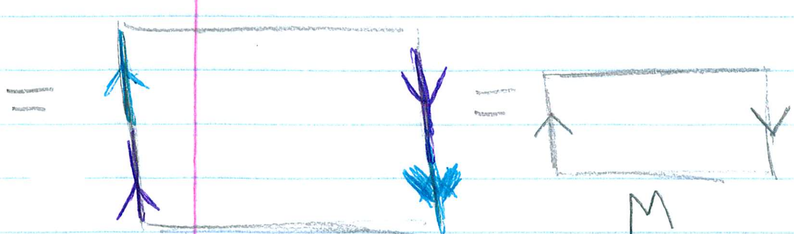
#④ It's an $\boxed{M}$



already supposed to be glued







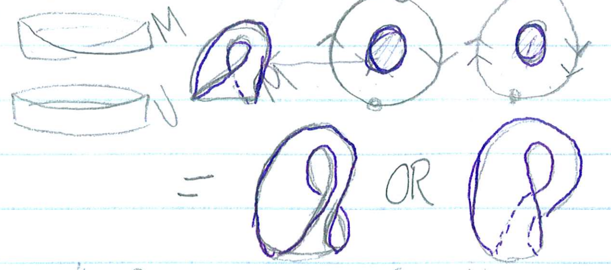
Outstanding Award again!

5 breaks :)

Sorry (we had only two today). My bad :)

Knowing what P^2 D^2 is what is $P^2 \# P^2$?

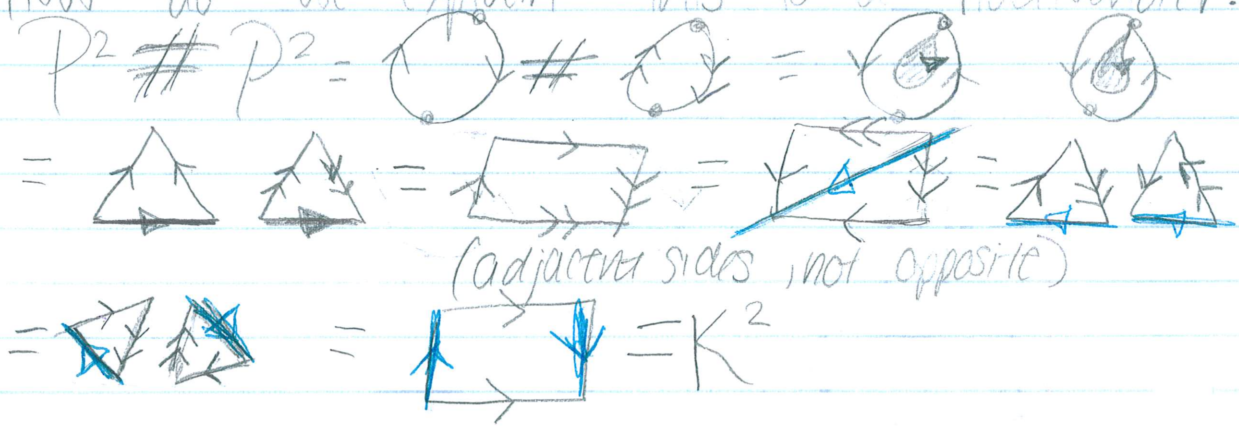
hint: Klein bottle



Summary: $P^2 \# P^2$
$= K^2$

How do we explain this to a flatlander?

$P^2 \# P^2 =$



(adjacent sides, not opposite)

$= K^2$

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Hannah

Hannah Bunow

REVIEW

$f^{-1}, f, a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, e, e$

$= abab \# cdfc^{-1}d^{-1}e^{-1}f^{-1}$

PI

Assumptions: #1 The sphere is made of special material so that it can pass through itself

#2 Every motion (deformation) is smooth (no sharp corners)

once stretched it creates a tight kink

very good notes!

the Geometry Center (no longer exists bc lack of \$) (UIUC)

William Thurston (-2012) (won Field's Medal)

"Outside In" (name of Movie)

mathmatical Nobel Prize

Smiles = frowns = total net Turning number

sadles are both frown & smile

opposite TN's means ∅ + inside out

waves avoid sharp bends (as long as TN's 1)

whole sphere is wavy

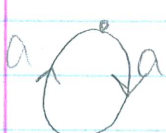
caps go through each other

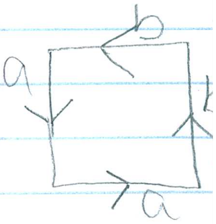
twist opposite directions to untwist guide strips

one side then goes from in to out

corigations never too deep

can twist in space bc it's S^2

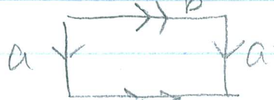
P^2  (aa)

$P^2 \# P^2 = aabb$ 

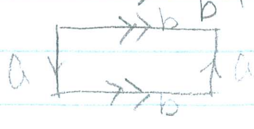
$(aa)\ (bb)$

(K^2)

S^2  (aa^{-1})

T^2  $(aa^{-1}bb^{-1})$

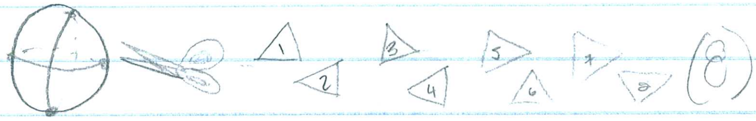
$T^2 \# P^2 = aba^{-1}b^{-1}cc$

K^2  $(abab^{-1})$

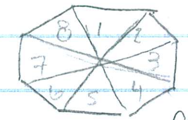
New Definition of a 2-D universe!

#1 no boundary
#2 connected
#3 locally a blackboard
#4 the space can be cut into finitely many triangles
→ topologists say it's $\boxed{\text{compact}}$ (replace one w/ the other)
(the universe we're looking at is finite)
(total space is finite)
(analogy: 4+3+1+2 = 10)

QUESTION: What are all the possible 2-D universes (satisfying #1-4)
Start w/ #4 → triangles



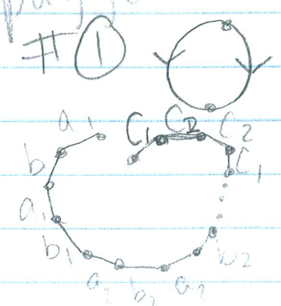
Glue the triangles to get a disc



contains all # but order doesn't matter

Statement: Then, it can be shown ... that can be transformed into another polygonal shape (cutting + pasting)

Possibilities: #1



When label switches it cannot be glued

$\underbrace{a_1 b a^{-1} b^{-1}}\underbrace{a_2 b_2 a_2^{-1} b_2^{-1}}, c_1 c_2 c_1 c_2$

$\underbrace{\text{Torus}(T^2) \# \text{Torus}(T^2)}_{m}, \# P^2$

m (depends on beginning surface)

continued on nxt pg

m \ n	0
0	S^2

one of m,n can be zero (0)

(11)

Hannah

useless table ✓

m/n	0	1	2 K^2	3 $K^2\#P^2$	4 $K^2\#K^2$
0	S^2	P^2	$T^2\#P^2$	$P^2\#P^2\#P^2$	$P^2\#P^2\#P^2\#P^2$
1	T^2				
2	$T^2\#T^2$	$T^2\#P^2$		$T^2\#T^2\#P^2\#P^2\#P^2$	
3	$T^2\#T^2\#T^2$	$T^2\#T^2\#P^2$			
4	$T^2\#T^2\#T^2\#T^2$				

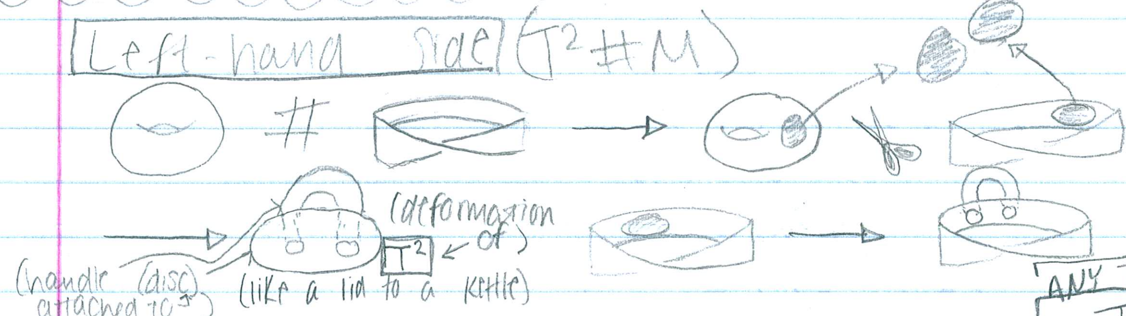
SAME

Where is K^2? ✓ — $T^2\#P^2 = K^2\#P^2$ $\quad$ ($P^2 \setminus D^2$ is M)

Where is $K^2\#P^2$? ✓ — $\overset{\text{Möbius band}}{\equiv}$ $T^2\#M = K^2\#M$

Where is $K^2\#K^2$? ✓ — $\equiv$ $T^2\#P^2 = P^2\#P^2\#P^2$

Left-hand side $(T^2\#M)$



ANY T^2 + P^2 COMBO

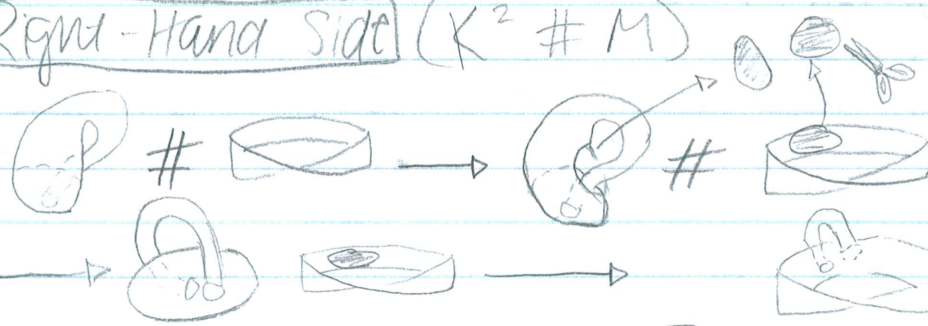
$T^2\#T^2\#P^2$

$= T^2\#P^2\#P^2$

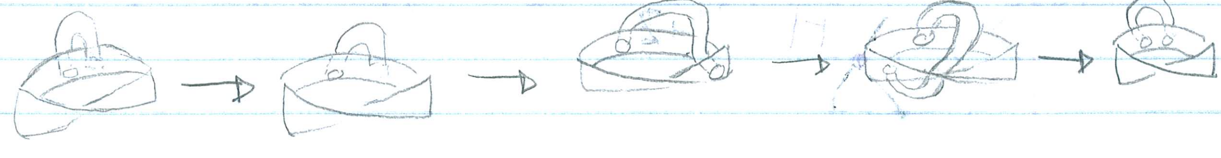
$= P^2\#P^2\#P^2\#P^2\#P^2$

CAN BE CONVERTED

Right-Hand Side $(K^2\#M)$



Here's How: (homeomorphism)



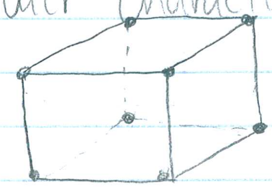
Finally: any 2-D universe satisfying #1-4 is either

only {
#① S^2
#② $T^2\#\ldots\#T^2$ $(m \geq 1)$ $\boxed{\text{OR}}$
#③ $P^2\#\ldots\#P^2$ $(n \geq 1)$

Topological invariant is a number (or something more complicated) which does not change under deformation ⟹

Topological invariant:
same space ⟹ same invariant
diff space ⟹ diff invariant
(contrapositive of the previous statement)
Invariant we use are called: Euler * Freud ☆

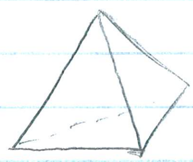
* Euler characteristic χ



V → # of vertices
E → # of edges
F → # of faces

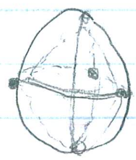
0-D	1-D	2-D

$\chi(S^2) = V - E + F$
$= 8 - 12 + 6$
$= 2$



$\chi(S^2) = V - E + F$
$= 4 - 6 + 4$
$= 2$

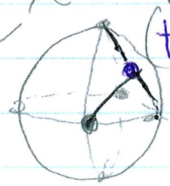
# of face characteristic



$\chi(S^2 = V - E + F$
$= 6 - 12 + 8$
$= 2$

why? χ invariant requires a "full proof": p.h.D. level mathematical understanding

change of $\chi = V - E + F$



$(+1) - (+3) + (+2) = 0$ it doesn't matter how you change this

① $\chi(S^2) = 2$
② $\chi(T^2 \# \ldots \# T^2) = V - E + F$
$\downarrow$ m
$= 1 - 2m + 1$
$= 2 - 2m$

③ $\chi(P^2 \# \ldots \# P^2) = V - E + F$
$\downarrow$ n
$= 1 - n + 1 = 2 - n$

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Hannah

CONCLUSION

Two Different Kinds of surfaces:

	S^2	T^2	$T^2 \# T^2$
χ:	2	0	-2

ORIENTABLE (no M)

Surfaces (cont):	$T^2 \# T^2 \# T^2$
(cont) χ:	4

NON-ORIENTABLE (M):

(M):	P^2	$P^2 \# P^2$	$P^2 \# P^2 \# P^2$
χ:	1	0	-1

(M):	$P^2 \# P^2 \# P^2 \# P^2$
χ:	-2

HW: χ (surface orientable) = -4026, what is this surface?

$2014\,T^2$

$\chi(S^2) = 2$

$\chi(\underbrace{T^2 \# \dots \# T^2}_{m}) = 2 - 2m$

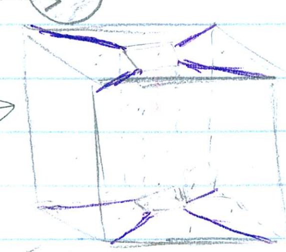
$\chi(\underbrace{P^2 \# \dots \# P^2}_{n}) = 2 - n$

$\chi(M) = V - E + F$

Euler characteristic | a 2-D universe surface

annulus 

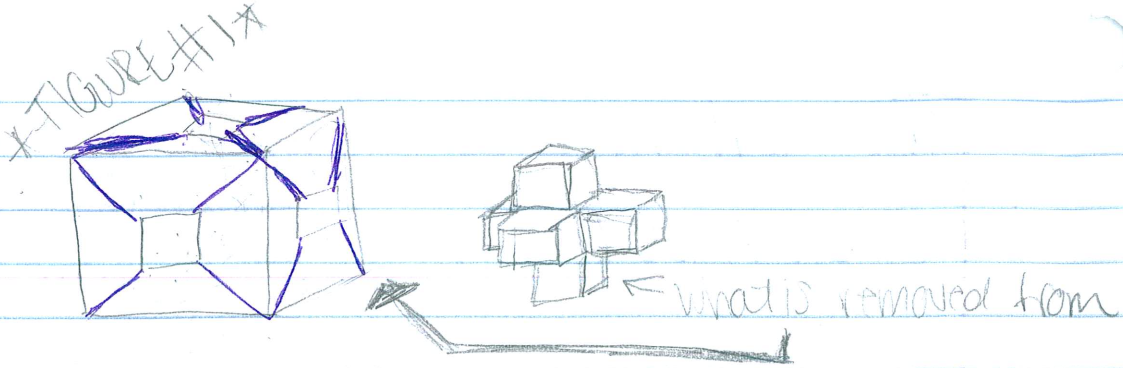
same as



$\chi(T^2) = V - E + F$

this works if m=1	$= 16 - (12+12+8) + (4+4+8)$
$\chi(T^2) = 2 - 2m$	$= 16 - 32 + 16$
$2 - 2 \cdot 1$	$= 0$
$2 - 2 = 0$	

"HW" Day 2:



Question: what is the surface of this shape?

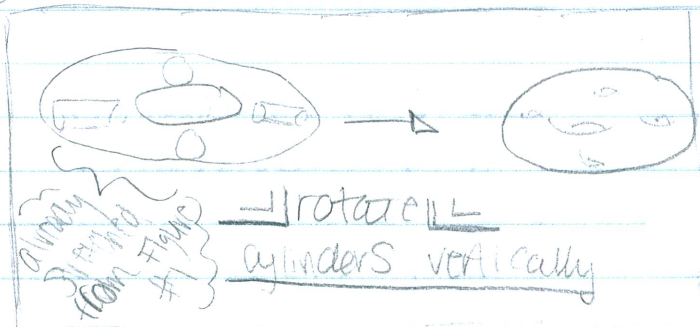
$\chi(M) = V - E + F$

$= (8+24+8) - (24+12+24+12+24) + (24+24)$

$= 40 - 96 + 48$

$= -56 + 48$

$= -8$



$\chi(M) = -8$

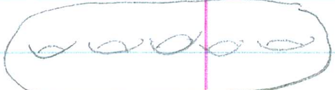
$-8 = 2 - 2m$

$-2 \quad -2$

$-10 = -2m$

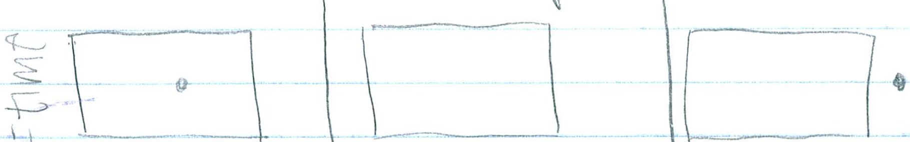
$-2 \quad -2$

$5 = m$

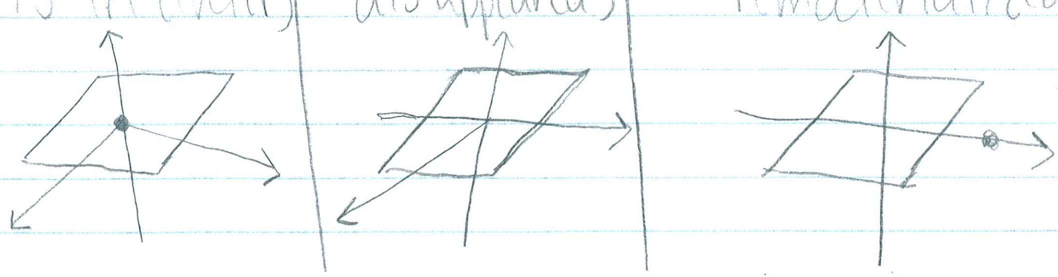


$(M) = 5T^2 \quad (T^2 \# T^2 \# T^2 \# T^2 \# T^2)$

4-D Shapes!!!



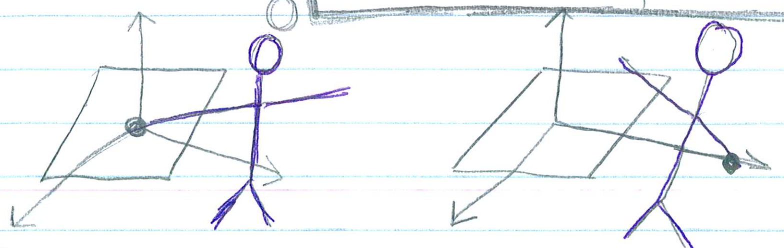
t = 0 (the point is in center) | t = 1 (point disappeared) | t = 2 (the point rematerialized)



14 Hannah

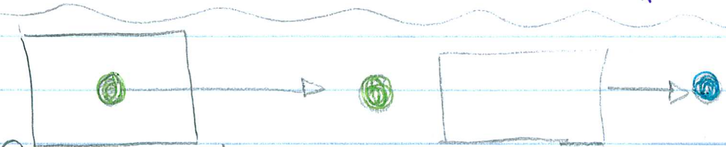
when drawing a 4-D on 2-D many things get lost

EX. #1:



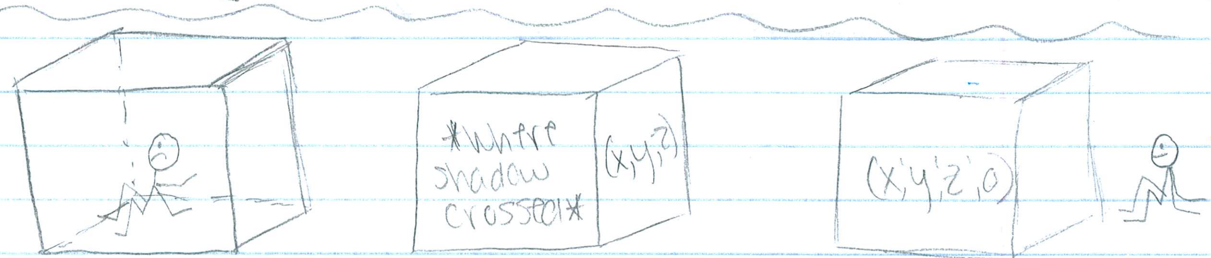
(x, y, 0)

EX. #2:



(green) SHADOW! restores to it's natural color (blue)

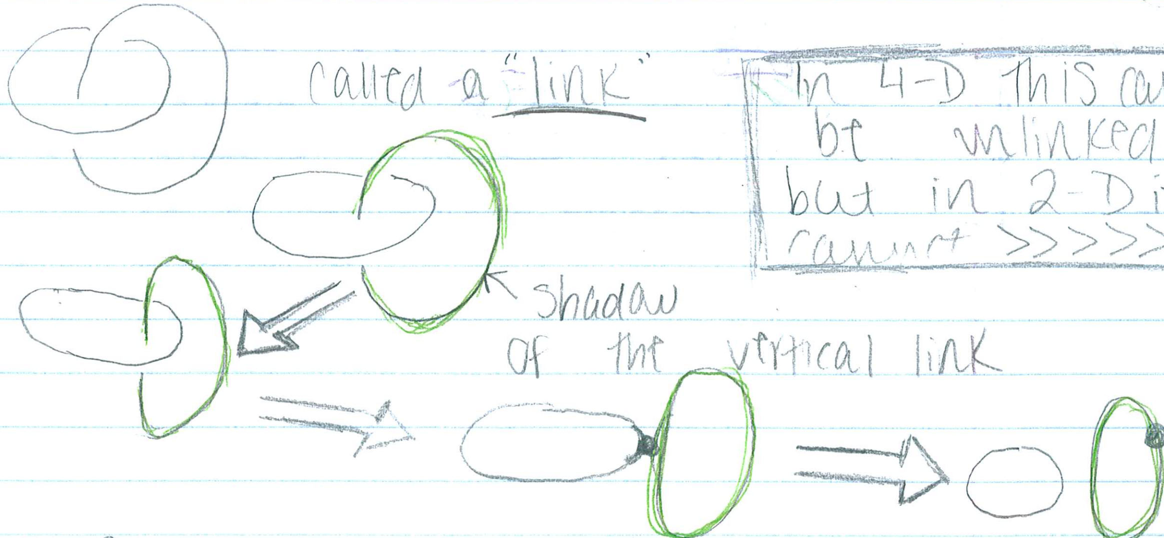
EX. #3:



t=0 t=1 t=2

a 4-D (person) picked him up and placed him somewhere in his world, the person's original world

previous examples #1:



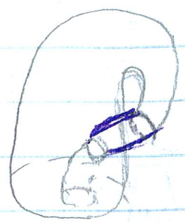
In 4-D this can be unlinked but in 2-D it cannot >>>>>

This seems strange because we can't see 4-D, we're using analogies, and this can only be described by math (to us): [Shadow can pass through but actual object cannot]

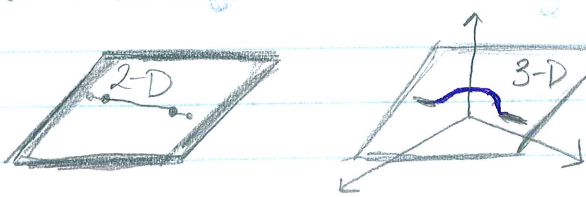
previous examples
X #2:

* actually has no intersection *

- This is the shadow of a Klein bottle
- Because it is in the 3-D world
- K^2 lives in a 4-D world



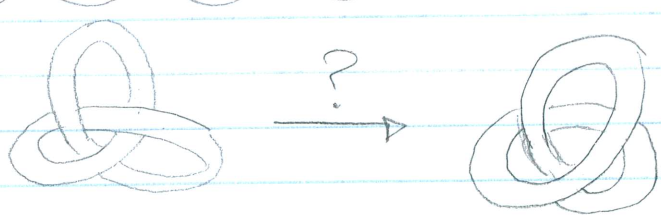
How:
(not broken)



How do we explain this to a flat (aram?)

not broken!

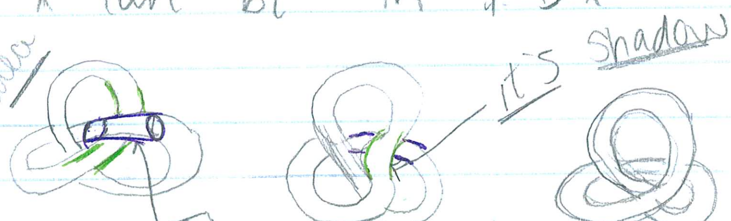
exercise:
X



* cannot happen in 3-D *
* can be in 4-D *
} How?!

mathematically called "surgery"

my idea



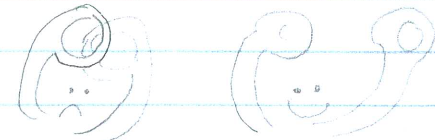
bends down through the shape which is a shadow

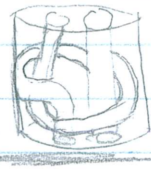
(15) Harmon

Coursera → register + take the courses
(website .org)
edX

PROJECT

#1: for people who like making things

PART I
build + draw + make 
K^2
make a series of clay models to show deformation
(so they can stand up also)

PART II
 *3-D* → isotopic 
series of pics to show deform.

people who like drawing

#2: for people who are interested in fashion

PART I: read the paper (7 pgs)
find topology elements in their design
2-D → T^2 + P^2
3-D → depends on geometry, 8 pieces
PART II: - try to design your own things
- explain the elements of it

fashion show

#3: Involves physics

for people who'd like to continue thinking about the universe shape question

#4: music + Möbius band
- explain connection → PART I (2 kinds)
- play a piece of music on keyboard - PART II

Simons Foundation (invariant of octave + modularity octave) (12 semi-tone chromatic scale)
- raised $ for math through business
- funds math research now (free time + free $)

pitch corresponds to frequency (CE + EC are =) but need to be same po[...]

(tritones) [harmony: every point on M corres. to a 2 note chord]

scales, repetition of individual notes

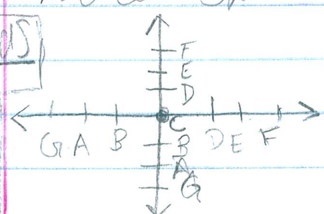
[where x = y → unison chords
otherwise → parallel chords]

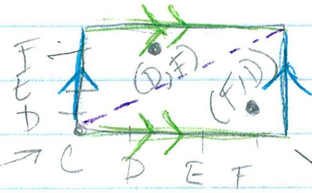
#5: ART
by Dali
finding topography within it
introduce Dali's work in general
analyze his art work
explain the special thing about it

#6: Create your own project
like writing a short story about these concepts to Flatlanders

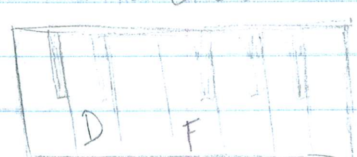
#4 cont.: the M in video is the space of all 2-note chords modulo octaves

identity sounds
symmetric with respect to the main diagonal

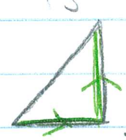




So, this is a torus..



When the shape is folded the blue line is glued to the green line like:



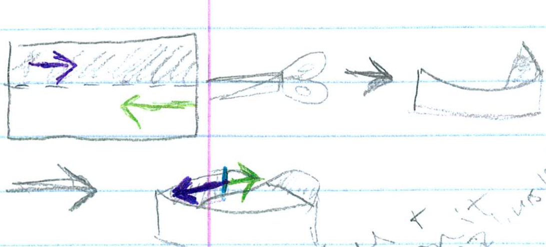
(which is an M)

Question: But why? (+ connect to standard M pic)

PART II of #4: the second connection
J.S. Bach → crab canon

(2nd musician plays the imitation piece backwards)

(play "same thing" @ diff times)



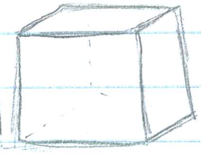
Take a song + #① make an M + write notes on it (write precisely) #② 2 musicians (1st) then you play the music

i.e. C D F G B A C →
← C A B G F D C
(crab canon)

(16)
Hannah

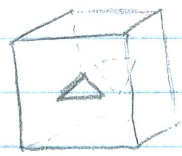
3-D Universes

review: T^3 glue 3 faces



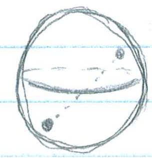
Conditions
#1 no boundary
#2 connected
#3 locally a classroom

$S^1 \times K^2$



glue front to back
with a left-right flip

The other 2 pairs are glued like those of T^3

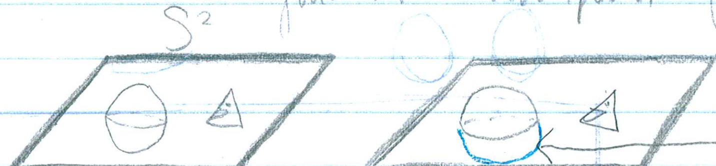


P^3 line that also goes through the center
↳ antipodal

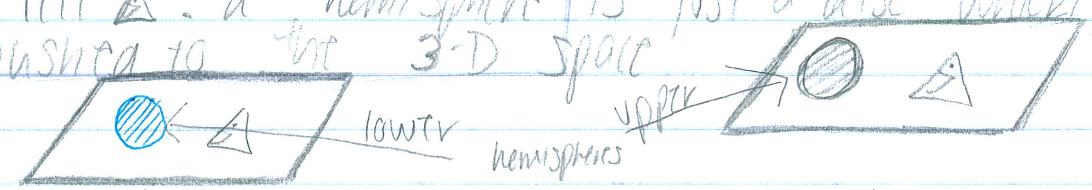
<<glue at antipodal points>> (but only do this for points on the surface)

How do we explain this to a flatlander?

S^2

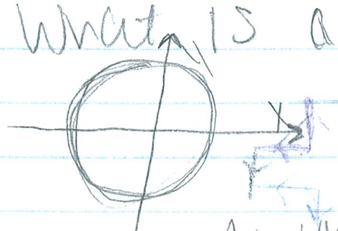


Tell △: a hemisphere is just a disc which is pushed to the 3-D space



Just glue along the discs the two hemispheres to get a sphere ← (the boundary circles)

Question: What is a hypersphere? (S^3)



Suppose the radius is 1

$\boxed{x^2+y^2=r^2 \quad r=\sqrt{x^2+y^2}}$

$(0,0)$
$x^2+y^2 \leq 1$

A unit disc is all the points on the plane whose distance to the origin is less than or equal to 1

A disc consists of all those (x,y)s

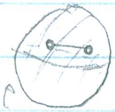
Make the projects a Story

Question: What is the analogous object to a disc in 3-D?

In 2-D 

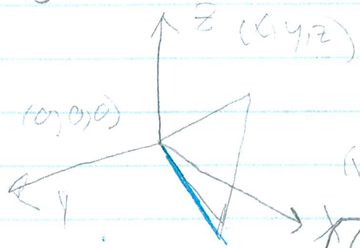
$\sqrt{x^2+y^2} \le 1$

- a disc

In 3-D 

- solid sphere

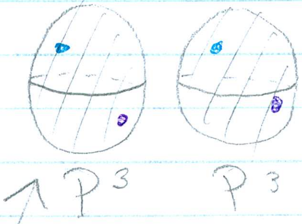
The distance between a point in the sphere is ≤ 1

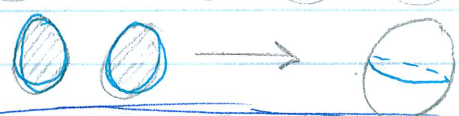


$\sqrt{\left(\sqrt{x^2+y^2}\right)^2+z^2}$

$=\sqrt{x^2+y^2+z^2}$ ← this is the simplest form for real numbers

ANALOGY





S^3 (hyper sphere)

↓

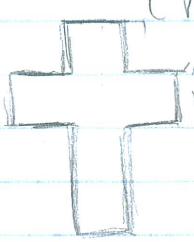
bounds a 4-D sphere

project #5 cont.

Corpus Hypercubus by Dali, Salvador

↓

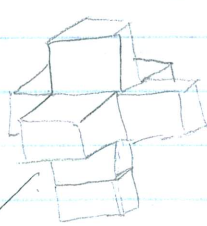
Crucifixion (real painting is in the MOMA)



"regular" cross

(can be folded into a cube)

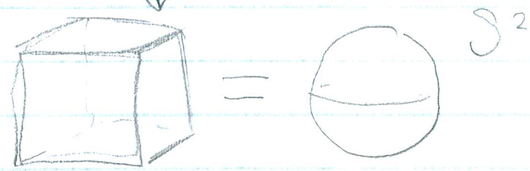
(called a net)



back cube

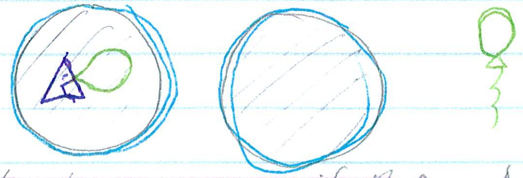
S^3 what should be done?

explain by using



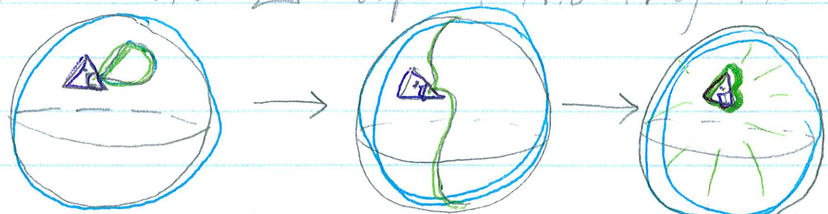
(17)
Hannah

Question/Exercise: S^2



∞ stretchable

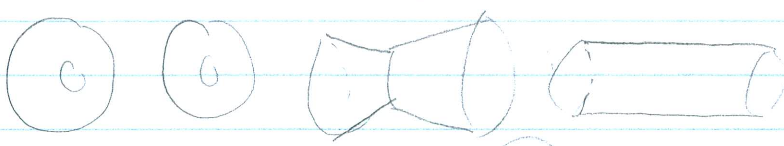
What happens if the $\triangle$ keeps inflating the balloon?



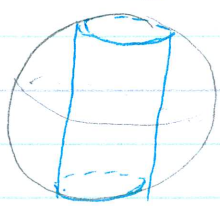
Conclusion: S^3 is different than the world we assume that we are in

Q: What if it kept going?

A: Same thing would happen. Eventually we'll be surrounded by balloon air.



Question/Exercise:



P^3

Is this cylinder a torus or a Klein bottle?

Outstanding Work! again

18 Hannah

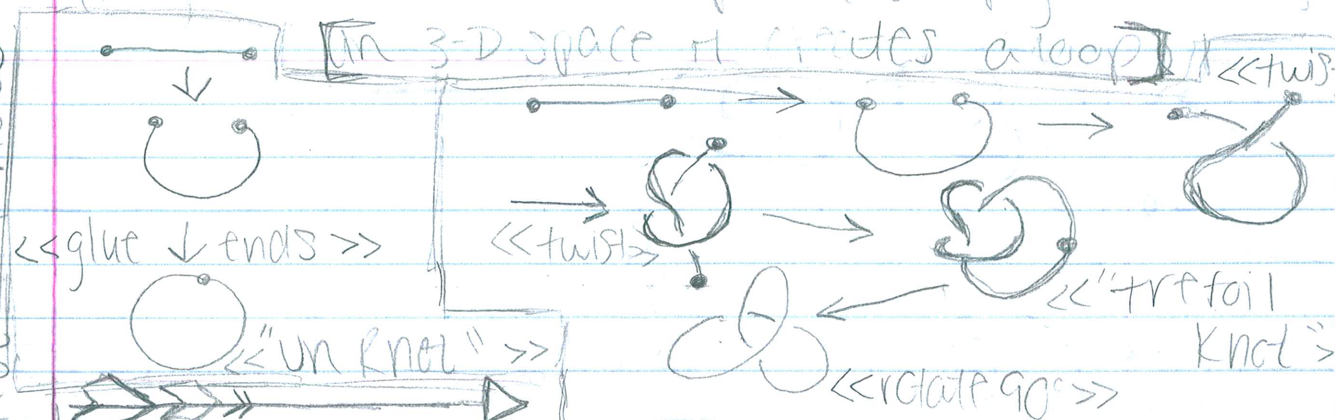
Hannah Burnau

1-D Universes in 3-D Space

(they are all homeomorphic (isotopy is tricky)

[in 3-D space it creates a loop]

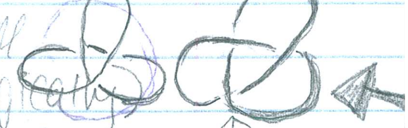
Knots

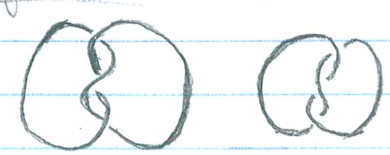


Trefoil Knots: the simpliest knots

It has been proved that these two trefoil knots are not isotopic. This question is considered impossible. Analogous to π, except far more complicated. We'll find some topological invariants for those two knots for comparison.

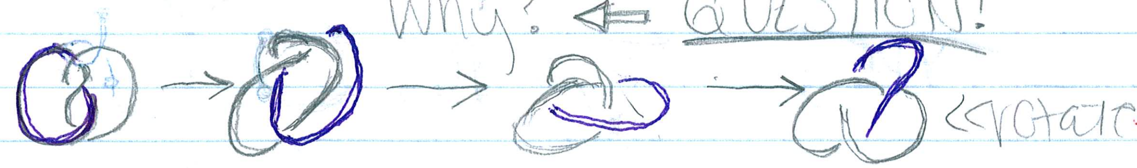
QUESTION: Are they the same or different? (isotopically) Can we deform one to the other?



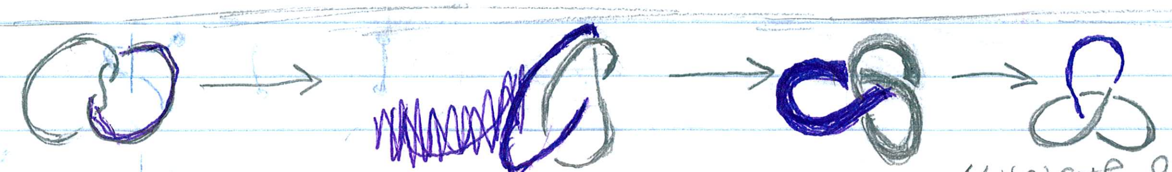


These are also knots, in fact, they are the same as

Why? ← QUESTION!



Exercise:

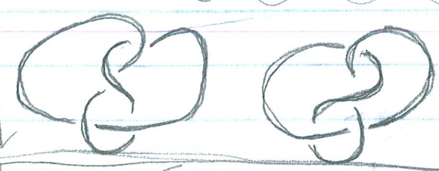


turn over →

FIGURE 8 KNOT

I like the font design

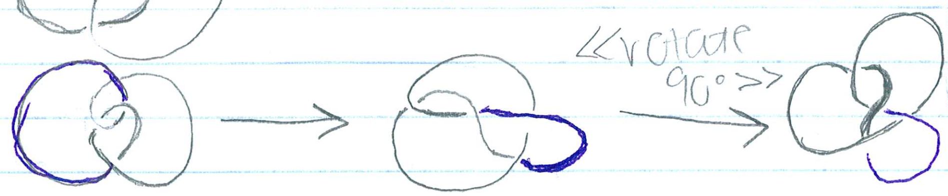
ISOTOPIC



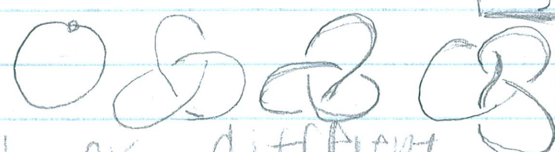
QUESTION: What's the relationship between these? Mirror Image

TRY. THIS. LATER.

EXERCISE:  also The figure 8



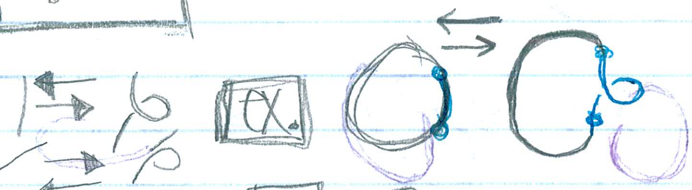
so far the way we use to study knots is to look at their 2-D diagrams on paper or on a blackboard;

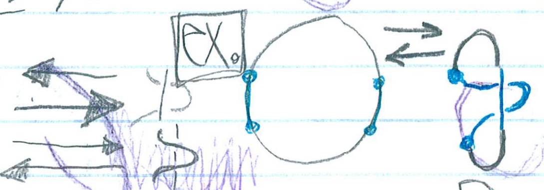
OUR GOAL: is to see why these knots →  are the same + or different.

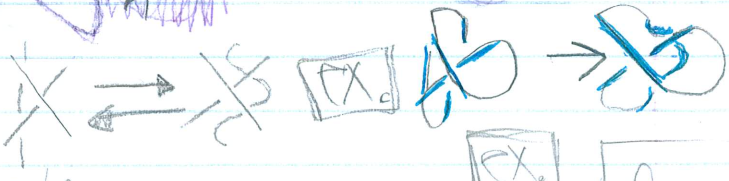
KNOT THEORY

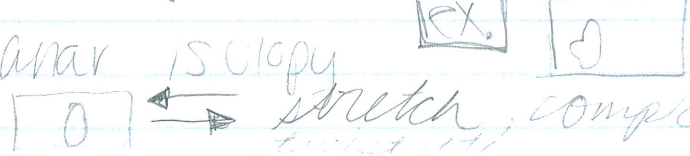
Reidemeister → German mathematician

MOVES:

TYPE I: 

TYPE II: 

TYPE III: 

TYPE Ø: planar isotopy 

stretch, compress, twist it!

(9) Hannah

This theorem is of theoretical value

THEOREM: (due to Reidemeister)

Two knots are isotopic if and only if one can be obtained from the other via a sequence of Reidemeister moves (Type 0, I, II, III) on their planar diagrams

#1 The "if" part is easier
#2 The "only if" part is proved by Reidemeister

EXPLCPt. 8 steps

We know  is isotopic to  according to the theorem on their planar diagrams one can be obtained from the other by a sequence of moves. How? (More steps; 8 total) But the theorems don't give a practical way to solve equations, (nature of math theorems)

Constructing a topological invariant

1st:

THE BRACKET POLYNOMIAL #1 $\langle O \rangle = 1$ [the BP of unknot is = 1

#2 $\langle \times \rangle = A \langle)(\rangle + \langle \asymp \rangle B$

turn over paper

$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$

erase unknot + add C

#3 $\langle O \cup L \rangle = C \langle L \rangle$ [O = unknot, ∪ = union
L = some other knot

#4 if A + B are planar isotopic
$\langle A \rangle = \langle B \rangle$ (Ex. $\langle O \rangle = \langle 8 \rangle$)

#2 to #3 is a 90° rotation

RULE REVIEW: #1 $\langle O \rangle = 1$ #4 $\langle O \cup L \rangle = C\langle L \rangle$

#2 $\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$

#3 $\langle \times \rangle = A\langle \asymp \rangle + B\langle)(\rangle$

Questions: What are A, B, C?
Why do we bother to introduce 3 variables instead of 1 in the first place?

Answers: This is bc we need more freedom so that we will be able to express the constraint imposed by the problem. (If our number of variables is too small, it's impossible that it'll lead to contradiction.)
Also we define the ~~third~~ variable by the first two.

Questions: What are the constraints?
Now:

Answer: invariance under moves of types 0, I, II, III

We want $\underset{[\text{left}]}{\langle \ \rangle} = \underset{[\text{right}]}{\langle \, || \, \rangle}$ [type II]

left $= A\langle \ \rangle + B\langle \ \rangle$

$= \boxed{A}\left(A\langle \ \rangle + B\langle \ \rangle\right) + \boxed{B}\left(A\langle \ \rangle + B\langle \ \rangle\right)$

$= \boxed{A^2}\langle = \rangle + \boxed{AB}\langle || \rangle + \boxed{BA}\underbrace{\langle O \cup = \rangle}_{\boxed{C\langle = \rangle}} + \boxed{B^2}\langle = \rangle$

$= (A^2 + ABC + B^2)\langle = \rangle + AB\langle || \rangle$

right $= \langle || \rangle$

20

Hannah

cloth / skin / pleather material
simple buttons
mini sewing kit

metal / silver or gold bendable material
some pearls
black stretchable cloth / material

giant wood beads?

scarf material
sewing kit

2 scarf ideas

Algebraic topology involves differences in knots

MORE RULES:
$\langle O \cup L \rangle = (-A^2 - A^{-2})\langle L \rangle$

$AB = 1 \longrightarrow B = \frac{1}{A} = A^{-1}$

$A^2 + ABC + B^2 = 0 \rightarrow A^2 + C + B^2 = 0$

$\Rightarrow C = -A^2 - B^2$

$= -A^2 - (A^{-1})^2 = -A^2 - A^{-2}$

Question: Is our polynomial the invariant under move type III?

$\langle \;\; \rangle = A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$

$= A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$ } type II move

[applying the rules BACKWARDS] $= A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$

$= \langle \;\; \rangle$

Answer: YES, it is invariant under type III moves

Question: What about under Type I

Answer: NO [bc:

$\langle \;\; \rangle = A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$

$= A\langle \;\; \rangle + A^{-1}(-A^2 - A^{-2})\langle \;\; \rangle$

$= (\cancel{A} - \cancel{A^{-1+2}} - A^{-1-2})\langle \;\; \rangle$

$= -A^{-3}\langle \;\; \rangle$]

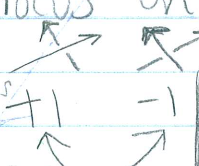
EVEN MORE RULES } Similarly $\langle \;\; \rangle = -A^{-3}\langle \;\; \rangle$

$\langle \;\; \rangle = -A^{3}\langle \;\; \rangle$

To fix this we use "Writhe number" "W"

(you can traverse loops)

"T"  [Focus on the crossings]



[to determine which is which ... just trace them rotations clockwise or counterclockwise]

$w(T)$ = sum of all numbers at a crossing

$= -(1) + (-1) + (-1)$

$= -3$

turn over page

peter pan collar ??? trefoil knot pencil sharpener

unknot $P(O) = -A^3(W(O)) \langle O \rangle = 1$

$P(O) = 1$ $\quad 1 \cdot 1 = 1$

The "W" doesn't depend on the direction we choose!

+1 change direction +1 [align w/ back arrow]

$W = 0$ { invariant under II, III, etc

Kaufman polynomial
(professor @ University of Chicago)

memorize: $P(K) = (-A^3)^{-W(K)} \langle K \rangle$

Kaufman polynomial — writhe number of the given knot

INVARIANT UNDER ALL 4 MOVES

Computation: $P(T)$

#1 $W(T) = -3$

#2 $\langle T \rangle = \langle \quad \rangle$ [apply rules]

$= A\langle \quad \rangle + A^{-1}\langle \quad \rangle$

$= A(-A^3)\langle \quad \rangle$

$+ A^{-1}(A\langle \quad \rangle + A^{-1}\langle \quad \rangle)$

$= A(-A^3)(-A^3)\langle O \rangle + A^{-1}(A(-A^3)\langle O \rangle$

$+ A^{-1}(-A^{-3})\langle O \rangle)$

$= A^{1+3+3} - A^{-1+1+3} - A^{-1-1-3}$

$= A^7 - A^3 - A^{-5}$

[itself isn't a topological invariant]

$P(T) = (-A^3)^{-W(T)} \langle T \rangle$

$= (-A^3)^3 (A^7 - A^3 - A^{-5})$

$= -A^9(A^7 - A^3 - A^{-5})$

[this is the Kaufman polynomial] $= -A^{16} + A^{12} + A^4$ → for

$\boxed{\text{for} \quad \text{it's } P \text{ is } -A^{-16} + A^{-12} + A^{-4}}$

$\boxed{\langle O \rangle = 1}$

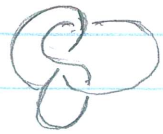
- diff. tref. knots
- diff from unknot

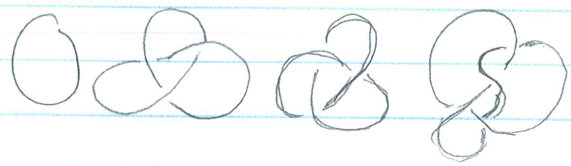
!

DIFFERENT

21

Hannah

P() $= A^{8} - A^{4} + 1 - A^{-4} + A^{-8}$

So . . .  are all diff bc of Kauffman's polynomial

Dear Hannah,

It's been a great pleasure to have you in class. You're the smartest and most mature. I also learned how to take better notes from you. Look forward to your final project presentation!