AMHERST COLLEGE

Department of Mathematics

COMPREHENSIVE EXAMINATION

Multivariable Calculus and Linear Algebra

2:00 pm Friday, February 1, 2013

Seeley Mudd 204/206

There are 8 problems (10 points each, totaling 80 points) on this portion of the examination. Record your answers in the blue book provided. **Show all of your work.**

1. Evaluate the following integrals:

   (a) $\displaystyle\iint_R (x+y)\,dy\,dx$, where R is the top half of the circle of radius 2 centered at the origin.

   (b) $\displaystyle\int_C (2xy+\tan(x^3))\,dx + \big(x^2+2xy)\,dy$, where C is the closed curve that begins at $(0,0)$, then follows $y=x^2$ to $(1,1)$, next follows $y=1$ to $(0,1)$, and finally follows $x=0$ back to $(0,0)$. .

2. Compute the volume of the 3-dimensional region that lies above the paraboloid $z=x^2+y^2$ and inside the sphere $x^2+y^2+z^2=2$. Hint: Cylindrical coordinates.

3. Consider the function

$$f(x,y)=\begin{cases}\dfrac{x^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

   (a) Show that f is continuous at $(0,0)$.

   (b) Find $f_x(0,0)$ and $f_y(0,0)$.

4.

   Find the maximum and minimum values of the function $f(x,y)=6x+4y$ subject to the constraint $3x^2+2y^2=5$.

   Find the maximum and minimum values of the function $f(x,y)=x^2+4y$ subject to the constraint $x^2+2y^2=8$.

   Find the maximum and minimum values of the function $f(x,y)=2x^3+12y$ subject to the constraint $x^2+y^2=2$.

5. Let A be a 5×7 matrix with real entries. Answer the following questions about A and briefly justify your answers:

   (a) Show that the nullspace of A has dimension at least 2.

   (b) Assume that the nullspace has dimension exactly 2. What does this imply about the column space of A?

6. Let U and W be subspaces of a vector space V. Define

$$U + W = \{u + w \mid u \in U,\ w \in W\}.$$

   (a) Prove that $U + W$ is a subspace of V.

   (b) When $V = \mathbb{R}^2$, give an example where $U \cup W$ is not a subspace of $\mathbb{R}^2$.

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (a) Determine the eigenvalues of A and a basis of each eigenspace of A.

   (b) Is A diagonalizable? Explain your reasoning.

8. Let $T : V \to W$ be a linear transformation from the vector space V to the vector space W. Assume that $v_1, ..., v_n \in V$ have the property that $T(v_1), ..., T(v_n)$ are linearly independent.

   (a) Prove that $v_1, ..., v_n$ are also linearly independent.

   (b) Assume in addition that $v_1, \ldots, v_n$ span V. Prove that T is one-to-one (injective).

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Comprehensive Examination: Algebra

Friday, February 1, 2013

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G be a group, let $H \subseteq G$ be a subgroup, and let $N \triangleleft G$ be a **normal** subgroup. Define
$$NH = \{xh : x \in N \text{ and } h \in H\}.$$
Prove that NH is a subgroup of G.

2. **(25 points).** Let G_1 and G_2 be groups, let $\phi : G_1 \to G_2$ be a homomorphism, and let $H_1 \subseteq G_1$ be a subgroup. Recall that the set
$$H_2 = \{\phi(x) : x \in H_1\}$$
is a subgroup of G_2, called the *image of H_1 under ϕ*, sometimes notated $\phi(H_1)$.
   (a) (10 points). If G_2 is finite, prove that $|H_2| \Big| |G_2|$.
   That is, prove that the order of H_2 divides the order of G_2.
   (b) (15 points). If G_1 is finite, prove that $|H_2| \Big| |G_1|$.
   That is, prove that the order of H_2 divides the order of G_1.

3. **(25 points).** Consider the group S_{100} of permutations of the set $\{1, 2, 3, \dots, 100\}$. Let $\sigma \in S_{100}$ be the permutation
$$\sigma = (3\ 6\ 4)(1\ 5\ 2\ 4)(1\ 6\ 5\ 3\ 2).$$
   (a) (7 points). Write σ as a product of **disjoint** cycles.
   (b) (7 points). Compute the **order** of σ.
   (c) (8 points). For each integer $n = 7, 8, \dots, 100$, let $\tau_n = (1\ n\ 5)$. For each such n, decide whether the product $\sigma\tau_n$ is an **even** or **odd** permutation.

4. **(25 points).** Let R be a ring.
   (a) (10 points). Define what it means for a subset $I \subseteq R$ to be an **ideal** of R. If you use any other technical terms like "closed," "subring," "subgroup," "coset," etc., you must fully define those terms as well.
   (b) (15 points). Suppose that R is commutative and has a multiplicative identity 1. Let $I \subseteq J \subseteq R$ be ideals, and suppose that the quotient ring R/I is a field.
   If $I \subsetneq J$, prove that $1 \in J$.
   [In fact, it is a Theorem from Math 350 that $J = R$ in this case, but you are only being asked to prove that $1 \in J$. In particular, however, you may **not** quote the $J = R$ theorem.]

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Comprehensive Examination: Analysis
February 1, 2013

Answer all four problems in the blue book provided. PLEASE SHOW ALL YOUR WORK.

1. (a) (2 points) What does it mean for a sequence (a_n) of real numbers to be *bounded*?

   (b) (2 points) State the Bolzano-Weierstrass Theorem as it applies to a bounded sequence (a_n) of real numbers.

2. Consider the sequence $(f_n)_{n\geq 1}$ of functions where $f_n : [0, \infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 - nx & \text{for } 0 \leq x \leq \frac{1}{n}; \\ 0 & \text{for } x > \frac{1}{n}. \end{cases}$$

   (a) (8 points) Prove that (f_n) converges pointwise to a function f and give an explicit description of f.

   (b) (6 points) Prove that (f_n) does not converge uniformly to f.

3. (a) (4 points) State the Cauchy Criterion for a series $\sum_{n=1}^{\infty} a_n$ of real numbers to converge.

   (b) (8 points) Suppose that a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges absolutely. Prove that the series converges.

4. (10 points) For this question, do EITHER part (a) OR part (b), NOT BOTH.

   (a) Prove that the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}; \end{cases}$$

   is not Riemann-integrable.

   (b) Prove that a nonempty compact set K of real numbers has a maximum element: that is, show that there is $x \in K$ such that $x \geq y$ for all $y \in K$.