AMHERST COLLEGE

Department of Mathematics

COMPREHENSIVE EXAMINATION

Multivariable Calculus and Linear Algebra

2:00 pm Friday, January 31, 2014

Seeley Mudd 204, 205, and 206
There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. Record your answers in the blue book provided. Show all of your work.

1. Find the critical points of the function \( f(x, y) = x^4 - 4xy + 2y^2 \) and classify as a local maximum, a local minimum, or a saddle point.

2. Suppose the plane \( z = 2x - y - 1 \) is tangent to the graph of \( z = f(x, y) \) at \( P = (5, 3) \).
   
   (a) Determine \( f(5, 3) \), \( \frac{\partial f}{\partial x}(5, 3) \), and \( \frac{\partial f}{\partial y}(5, 3) \). 
   
   (b) Estimate \( f(5.2, 2.9) \).

3. Calculate the volume of the region inside the sphere \( x^2 + y^2 + z^2 = a^2 \) and outside the cylinder \( x^2 + y^2 = b^2 \), where \( a > b \), by using an appropriate double integral.

4. Suppose that \( \mathbf{r}(t) = (3\sqrt{2}t, e^{-3t}, e^{3t}) \) describes the position of an object at time \( t \).
   
   (a) Calculate the acceleration of the object at time \( t \).
   
   (b) Calculate the speed of the object at time \( t \). Simplify by factoring the expression under the square root.
   
   (c) Calculate the total distance traveled by the object between times \( t = 0 \) and \( t = 1 \).

5. Consider the function 
   
   \[
   f(x, y) = \begin{cases} 
   \frac{3y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
   0 & \text{if } (x, y) = (0, 0). 
   \end{cases}
   \]

   (a) Show that \( f \) is continuous at \((0,0)\).
   
   (b) Find \( f_x(0, 0) \) and \( f_y(0, 0) \).

6. Suppose \( T : V \rightarrow V \) is a linear transformation, \( \mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \} \) is a basis for \( V \), and the matrix representation of \( T \) with respect to \( \mathcal{B} \) is

   \[
   A = \begin{bmatrix} 2 & 3 & 5 \\
   7 & 11 & -3 \\
   -1 & 19 & 0 \end{bmatrix}.
   \]

   Determine \( T(2\mathbf{b}_1 + 4\mathbf{b}_3) \) as a linear combination of \( \mathbf{b}_1, \mathbf{b}_2, \) and \( \mathbf{b}_3 \).
7. Let \( A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix} \).

(a) Compute the eigenvalue(s) of \( A \).

(b) Find an invertible matrix \( C \) such that \( C^{-1}AC \) is diagonal.

8. Let \( A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & 4 & \alpha \end{bmatrix} \), where \( \alpha \) is some real number.

(a) For what values of \( \alpha \) does the equation \( Ax = b \) have at least one solution for all \( b \in \mathbb{R}^3 \)?

(b) For the remainder of the problem set \( \alpha = 11 \). Find the general solution to \( Ax = 0 \).

9. Suppose \( \{u, v\} \) is a basis for a vector space \( V \). Prove that \( \{u + 2v, 3u - v\} \) is also a basis for \( V \).
1. (20 points). Let $G$ and $H$ be groups, let $\phi : G \to H$ be a homomorphism, and suppose $g \in G$ is an element of some finite order $n \geq 1$.
   
   (a) (10 points). Show that that order of $\phi(g)$ divides $n$.
   
   (b) (10 points). Suppose that $|G| = 200$, $|H| = 72$, and the chosen element $g \in G$ has order $n = 25$. Prove that $g$ belongs to the kernel of $\phi$.

2. (30 points). Consider the group $S_{10}$ of permutations of the set $\{1, 2, 3, \ldots, 10\}$. Let $\sigma, \tau \in S_{10}$ be the permutations $\sigma = (1, 2, 3)(4, 5, 6)$ and $\tau = (3, 4)(2, 7, 8, 5)$.
   
   (a) (6 points). Write $\sigma \tau$ as a product of disjoint cycles.
   
   (b) (12 points). Compute the order of each of $\sigma$, $\tau$, and $\sigma \tau$.
   
   (c) (12 points). Decide whether each of $\sigma$, $\tau$, and $\sigma \tau$ is an even or odd permutation; don’t forget to justify.

3. (25 points). Let $R$ be a ring.
   
   (a) (10 points). Define what it means for a subset $I \subseteq R$ to be an ideal of $R$. If you use any other technical terms like “closed,” “subring,” “subgroup,” “coset,” etc., you must fully define those terms as well.
   
   (b) (15 points). For the polynomial ring $R = \mathbb{R}[x]$, define $I = \{f \in R : f(2) = f(5) = 0\}$.
   
   Prove that $I$ is an ideal of $R$.

4. (25 points). A nonzero element $a$ of a ring is said to be nilpotent if there is a positive integer $n \geq 1$ such that $a^n = 0$. (The element 0 itself is not said to be nilpotent.)
   
   Let $R$ be a commutative ring, and let $I \subseteq R$ be an ideal. Prove that the following two statements are equivalent.
   
   (a) The quotient ring $R/I$ contains no nilpotents.
   
   (b) For every element $b \in R$ such that $b^m \in I$ for some positive integer $m \geq 1$, we have $b \in I$. 
Instructions: Do all five of the following problems. Write your solutions and all scratchwork in the blue book(s) provided. Show all of your work, and justify your answers.

1. (6 points)
   (a) State the Bolzano-Weierstrass Theorem for sequences of real numbers.
   (b) Give an example of a sequence that does not have a convergent subsequence.

2. (4 points) Find all values of $x$ for which the following series converges. Justify your answer.
   $$
   \sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n2^n}
   $$

3. (10 points) Use induction to prove that
   $$
   n! > 2^n
   $$
   for all positive integers $n$ greater than or equal to 4.

4. (10 points)
   (a) Let $f$ be a real-valued function defined on $\mathbb{R}$. State the $\varepsilon$-$\delta$ definition of what it means for $f$ to be continuous at a point $c$.
   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c$ and that $\{x_n\}$ is a sequence of real numbers that converges to $c$. Prove that the sequence $\{f(x_n)\}$ converges to $f(c)$.

5. (10 points) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $n \in \mathbb{N}$.
   (a) State the function $f$ to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.
   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.