AMHERST COLLEGE

Department of Mathematics

# COMPREHENSIVE EXAMINATION

## Multivariable Calculus and Linear Algebra

2:00 pm Friday, January 31, 2014

Seeley Mudd 204, 205, and 206

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. Record your answers in the blue book provided. **Show all of your work.**

1. Find the critical points of the function $f(x, y) = x^4 - 4xy + 2y^2$ and classify as a local maximum, a local minimum, or a saddle point.

2. Suppose the plane $z = 2x - y - 1$ is tangent to the graph of $z = f(x, y)$ at $P = (5, 3)$.
   (a) Determine $f(5, 3)$, $\frac{\partial f}{\partial x}(5, 3)$, and $\frac{\partial f}{\partial y}(5, 3)$.
   (b) Estimate $f(5.2, 2.9)$.

3. Calculate the volume of the region **inside** the sphere $x^2 + y^2 + z^2 = a^2$ and **outside** the cylinder $x^2 + y^2 = b^2$, where $a > b$, by using an appropriate double integral.

4. Suppose that $\mathbf{r}(t) = \langle 3\sqrt{2}t,\ e^{-3t},\ e^{3t} \rangle$ describes the position of an object at time $t$.
   (a) Calculate the acceleration of the object at time $t$.
   (b) Calculate the speed of the object at time $t$. Simplify by factoring the expression under the square root.
   (c) Calculate the total distance traveled by the object between times $t = 0$ and $t = 1$.

5. Consider the function

$$f(x, y) = \begin{cases} \frac{3y^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) Show that $f$ is continuous at (0,0).
   (b) Find $f_x(0, 0)$ and $f_y(0, 0)$.

6. Suppose $T : V \to V$ is a linear transformation, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $V$, and the matrix representation of $T$ with respect to $\mathcal{B}$ is

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & -3 \\ -1 & 19 & 0 \end{bmatrix}.$$

   Determine $T(2\mathbf{b}_1 + 4\mathbf{b}_3)$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$.

7. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix}$.

   (a) Compute the eigenvalue(s) of $A$.

   (b) Find an invertible matrix $C$ such that $C^{-1}AC$ is diagonal.

8. Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & 4 & \alpha \end{bmatrix}$, where $\alpha$ is some real number.

   (a) For what values of $\alpha$ does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for all $\mathbf{b} \in \mathbb{R}^3$?

   (b) For the remainder of the problem set $\alpha = 11$. Find the general solution to $A\mathbf{x} = \mathbf{0}$.

9. Suppose $\{u, v\}$ is a basis for a vector space $V$. Prove that $\{u + 2v, 3u - v\}$ is also a basis for $V$.

AMHERST COLLEGE
Department of Mathematics
COMPREHENSIVE EXAMINATION: MATHEMATICS 350
January 31, 2014

*Instructions:* Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(20 points).** Let $G$ and $H$ be groups, let $\phi : G \to H$ be a homomorphism, and suppose $g \in G$ is an element of some finite order $n \geq 1$.

   (a) (10 points). Show that that order of $\phi(g)$ divides $n$.

   (b) (10 points). Suppose that $|G| = 200$, $|H| = 72$, and the chosen element $g \in G$ has order $n = 25$. Prove that $g$ belongs to the kernel of $\phi$.

2. **(30 points).** Consider the group $S_{10}$ of permutations of the set $\{1, 2, 3, \ldots, 10\}$. Let $\sigma, \tau \in S_{10}$ be the permutations

$$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (3, 4)(2, 7, 8, 5).$$

   (a) (6 points). Write $\sigma\tau$ as a product of **disjoint** cycles.

   (b) (12 points). Compute the **order** of each of $\sigma$, $\tau$, and $\sigma\tau$.

   (c) (12 points). Decide whether each of $\sigma$, $\tau$, and $\sigma\tau$ is an **even** or **odd** permutation; don't forget to justify.

3. **(25 points).** Let $R$ be a ring.

   (a) (10 points). Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$. If you use any other technical terms like "closed," "subring," "subgroup," "coset," etc., you must fully define those terms as well.

   (b) (15 points). For the polynomial ring $R = \mathbb{R}[x]$, define

$$I = \{f \in R : f(2) = f(5) = 0\}.$$

   Prove that $I$ is an ideal of $R$.

4. **(25 points).** A nonzero element $a$ of a ring is said to be *nilpotent* if there is a positive integer $n \geq 1$ such that $a^n = 0$. (The element 0 itself is *not* said to be nilpotent.)

   Let $R$ be a commutative ring, and let $I \subseteq R$ be an ideal. Prove that the following two statements are equivalent.

   (a) The quotient ring $R/I$ contains no nilpotents.

   (b) For every element $b \in R$ such that $b^m \in I$ for some positive integer $m \geq 1$, we have $b \in I$.

AMHERST COLLEGE
Department of Mathematics
COMPREHENSIVE EXAMINATION: MATHEMATICS 355
January 31, 2014

*Instructions:* Do all five of the following problems. Write your solutions and all scratchwork in the blue book(s) provided. **Show all of your work, and justify your answers.**

1. (6 points)

   (a) State the Bolzano-Weierstrass Theorem for sequences of real numbers.

   (b) Give an example of a sequence that does not have a convergent subsequence.

2. (4 points) Find all values of $x$ for which the following series converges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n 2^n}$$

3. (10 points) Use induction to prove that

$$n! > 2^n$$

   for all positive integers $n$ greater than or equal to 4.

4. (10 points)

   (a) Let $f$ be a real-valued function defined on $\mathbb{R}$. State the $\varepsilon$-$\delta$ definition of what it means for $f$ to be continuous at a point $c$.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c$ and that $\{x_n\}$ is a sequence of real numbers that converges to $c$. Prove that the sequence $\{f(x_n)\}$ converges to $f(c)$.

5. (10 points) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $n \in \mathbb{N}$.

   (a) State the function $f$ to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.

   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.