Integrable systems and S^1-actions: constructions and bifurcations

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Symplectic manifolds and integrable systems

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- Denote by $\mathcal{X}_f$ the Hamiltonian vector field of f, which satisfies

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- Flows of $\mathcal{X}_{f_1}, \ldots \mathcal{X}_{f_n}$ induce (local) $\mathbb{R}^n$-action.

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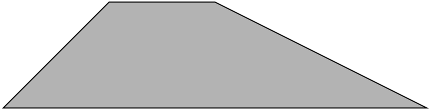
- i.e. a global Hamiltonian T^n-action.
- $F \colon M \to \mathbb{R}^n$, Atiyah, Guillemin-Sternberg (1982) showed that in this case $F(M)$ is the convex hull of the images of the fixed points.

Toric integrable systems: the classification



Theorem (Delzant, 1988)

Given any "Delzant polytope" $\Delta \subset \mathbb{R}^n$, there exists a unique (up to isomorphism) toric integrable system (M, ω, F) such that $F(M) = \Delta$.

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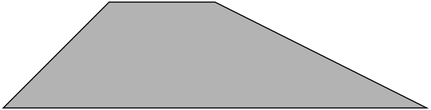

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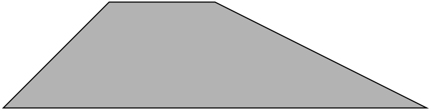

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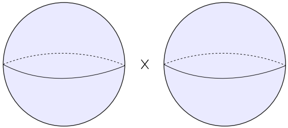
- $\{\text{toric systems}\} \overset{1-1}{\longleftrightarrow} \{\text{Delzant polytopes}\}$.
- Given a polygon, the associated system can be constructed by performing symplectic reduction on $\mathbb{C}^d$.

Example



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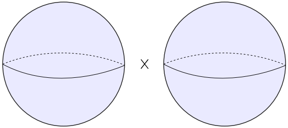

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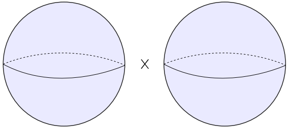

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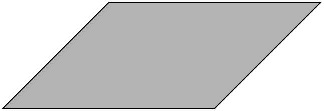

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- (M, ω, J) is a Hamiltonian S^1-space (as studied by Karshon, 1999).

Semitoric integrable systems: fibers

Points in simple semitoric systems:

- regular points;
- rank one: elliptic-regular points;
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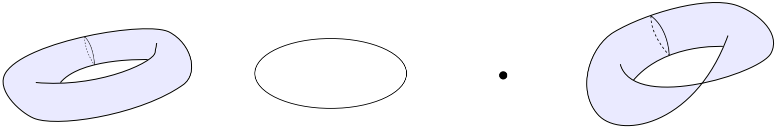

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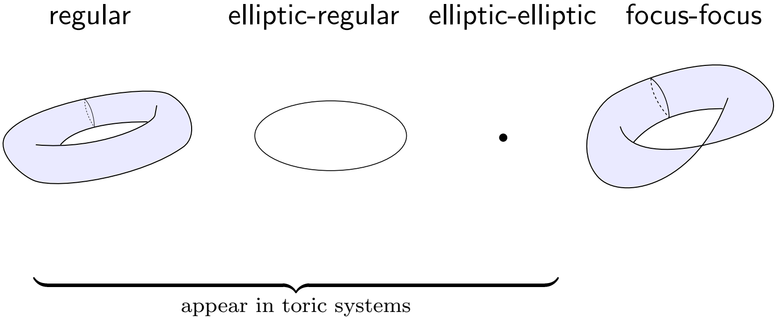

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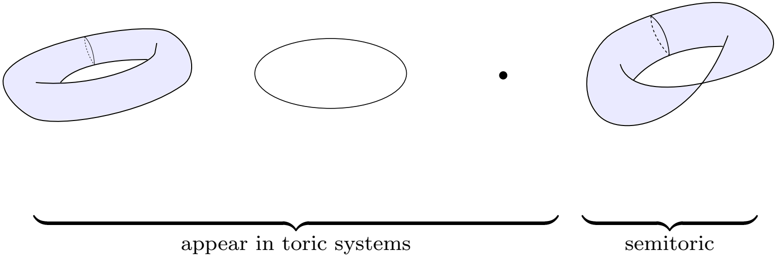

Semitoric integrable systems: classification

The five invariants:

(1) the number of focus-focus points invariant;

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(3) the height invariant;

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Theorem (Pelayo-Vũ Ngọc classification (2009, 2011))

1. *Two simple semitoric systems are isomorphic if and only if they have the same invariants (1)-(5);*
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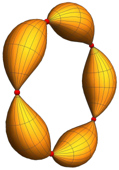

- In a non-simple semitoric system, several focus-focus points can lie on the same fiber.

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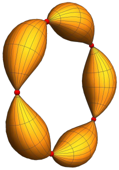

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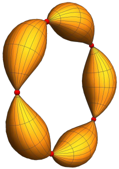

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General goal

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Goal

Given specified semitoric polygon invariant try to find an explicit system with that invariant (forgetting about the other invariants).

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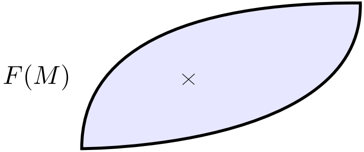

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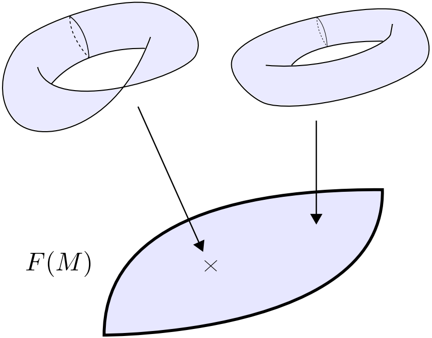

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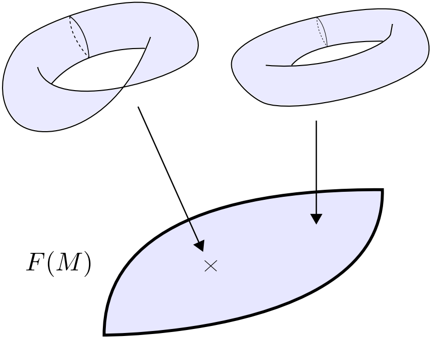

- Torus fibration $\to$ integral affine structure on $(F(M))_{\mathrm{regular}}$.
  - NOT equal to integral affine structure inherited from $\mathbb{R}^2$.

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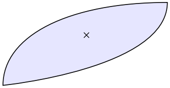

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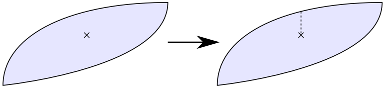

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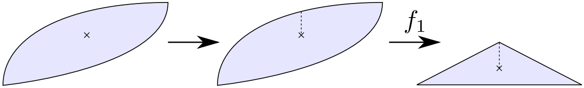

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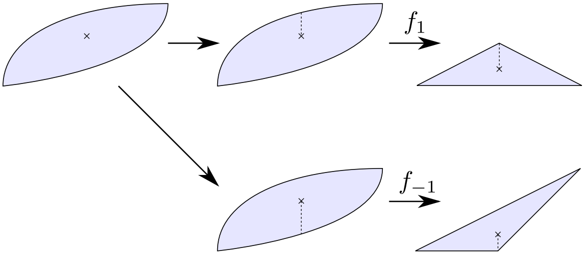

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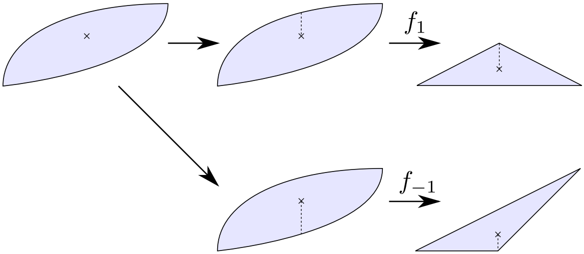

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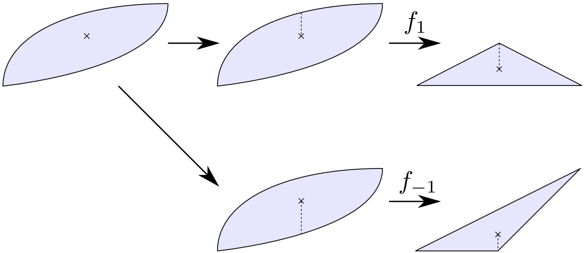

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- Marked semitoric polygon: includes information of invariants 1, 2, and 3.

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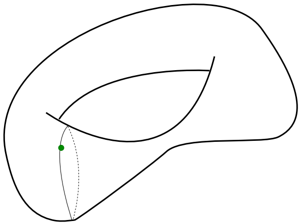

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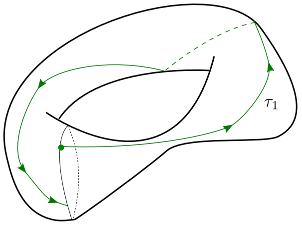

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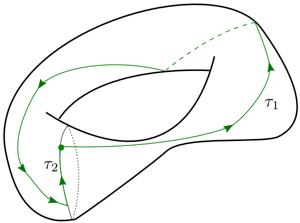

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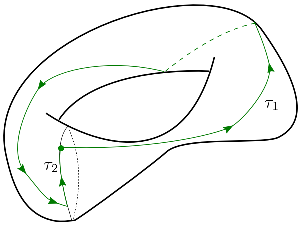
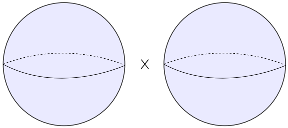


- There is an additional degree of freedom related to how this fiber sits with respect to the rest of the fibration, encoded in the twisting index invariant, an integer for each marked point in each choice of polygon.

Example: Coupled angular momenta

[Sadovskií and Zĥilinskií, 1999]



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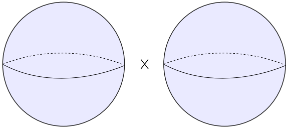

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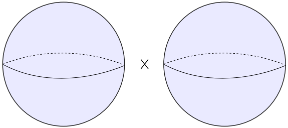

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- Notice J and H_0 generate S^1-actions

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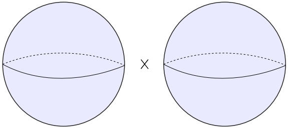

- $M = S^2 \times S^2, \quad \omega = R_1\omega_1 \oplus R_2\omega_2$
- coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$

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for $t \in [0,1]$ and $R_1 < R_2$.

- Notice J and H_0 generate S^1-actions
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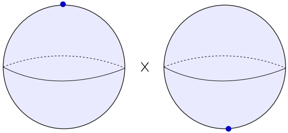

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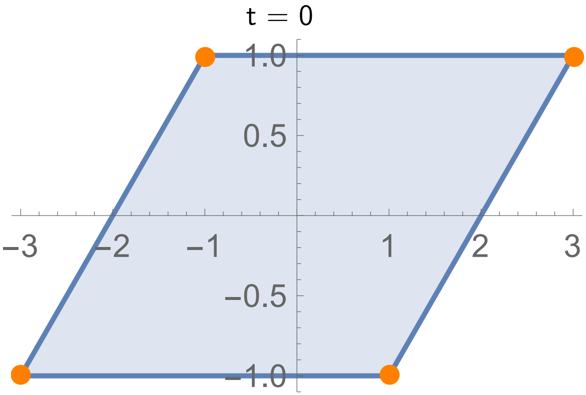
- In particular, $(J, H_{1/2})$ is semitoric.

Coupled angular momenta: moment map image



Semitoric with zero focus-focus points
(figure made in Mathematica)

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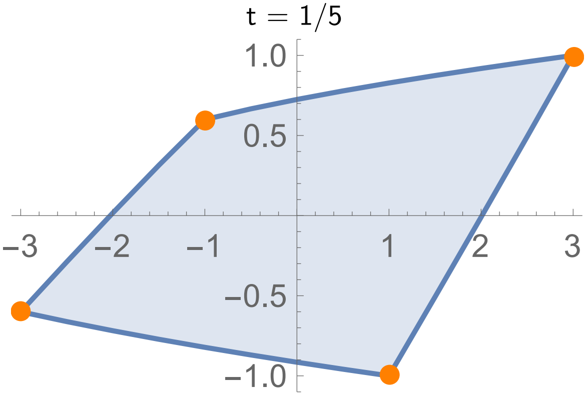

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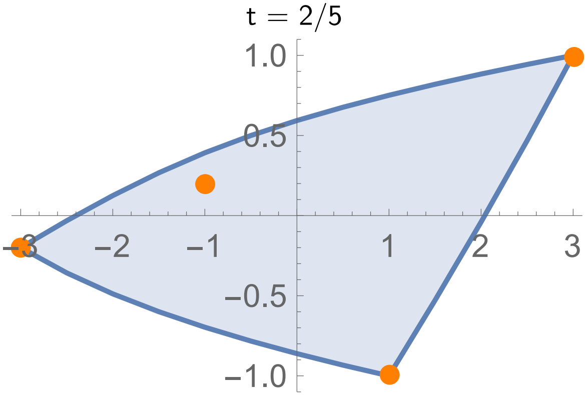

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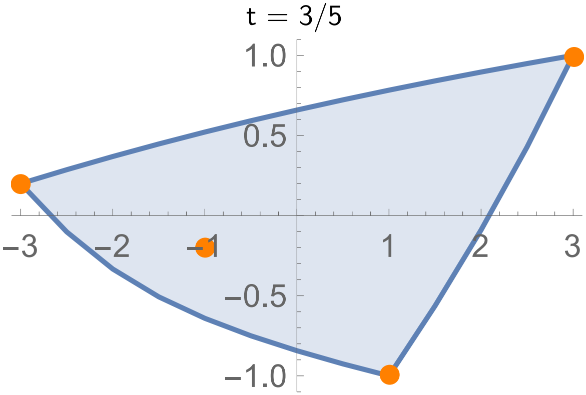

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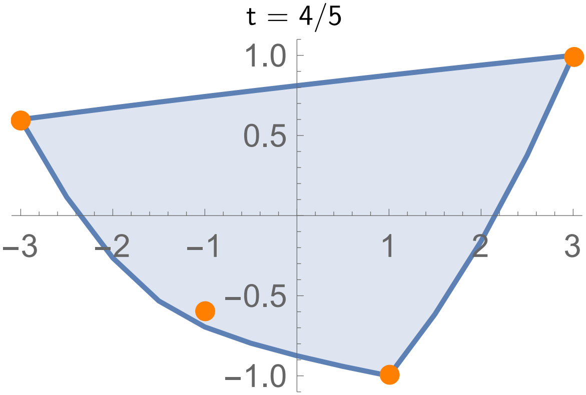

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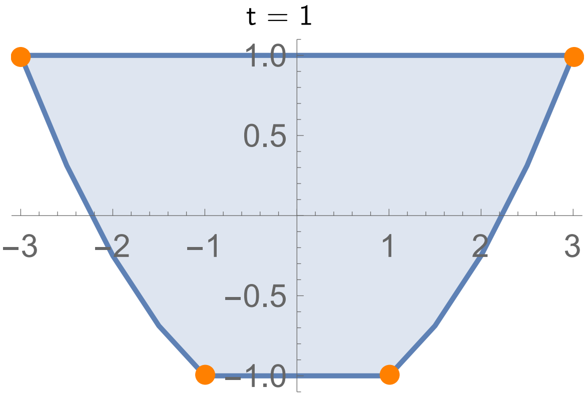

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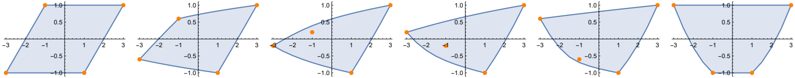
Coupled angular momenta: semitoric polygon

The image of the momentum map for (J, H_t):



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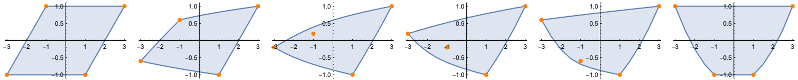

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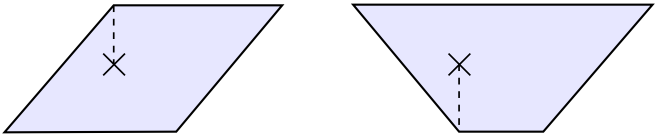

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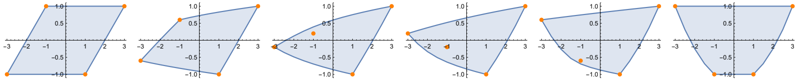

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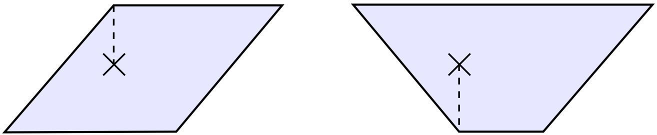

Idea

Interpolate between systems "related to the semitoric polygons" to find desired semitoric system.

Semitoric families: definition

Definition (Le Floch-P., 2018)

A semitoric family is a family of integrable systems (M, ω, F_t), $0 \leq t \leq 1$, where

- $\dim(M) = 4$;
- $F_t = (J, H_t)$;
- J generates an $\mathbb{S}^1$-action;
- $(t, p) \mapsto H_t(p)$ is smooth.
- it is semitoric for all but finitely many values of t (called the degenerate times).

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- *Semitoric families* (arXiv:1810.06915, to appear in *Memoirs of the AMS*)
- The behavior at the degenerate times can be very complicated!

Polygons in a semitoric family

- Invariance of polygon:

Lemma (Le Floch-P.)

The polygon invariant and number of focus-focus points in a (simple) semitoric family can only change at degenerate times.

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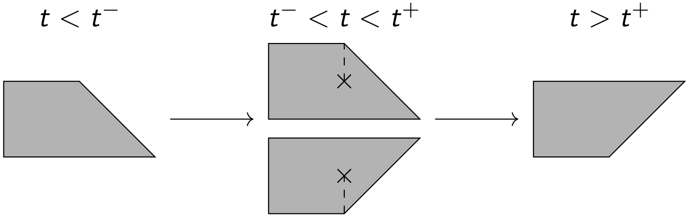
Lemma (Le Floch-P.)

Let $(M, \omega, (J, H_t))$ be a semitoric transition family with degenerate times t^- and t^+. Roughly, the set of semitoric polygons for $t^- < t < t^+$ is the union of the ones for $t < t^-$ and $t > t^+$.

Polygons in a semitoric family



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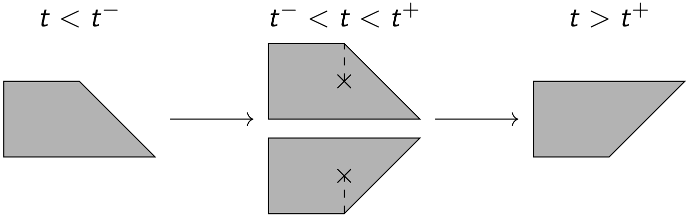
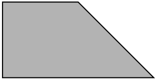


- For example, can we construct a system with the above polygons?

The first Hirzebruch surface



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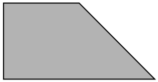

- Recall the first Hirzebruch surface, W_1, given by $\mathbb{C}^4$ reduced by Hamiltonian torus action:

$$N = (1/2)\left(|u_1|^2 + |u_2|^2 + |u_3|^2, |u_3|^2 + |u_4|^2\right) \text{ at } (2,1).$$

- Usual toric system: $J = 1/2|u_2|^2$, $H_0 = 1/2|u_3|^2$.

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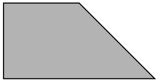

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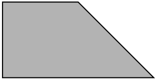

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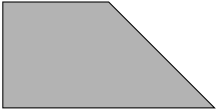

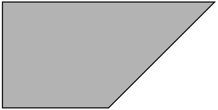

Example on W_1

Let $H_t = (1-t)H_0 + t(-H_0 + \gamma \mathrm{Re}(\overline{u}_1 u_3 \overline{u}_4))$.

Theorem (Le Floch-P.)

(J, H_t) is a semitoric transition family on W_1.

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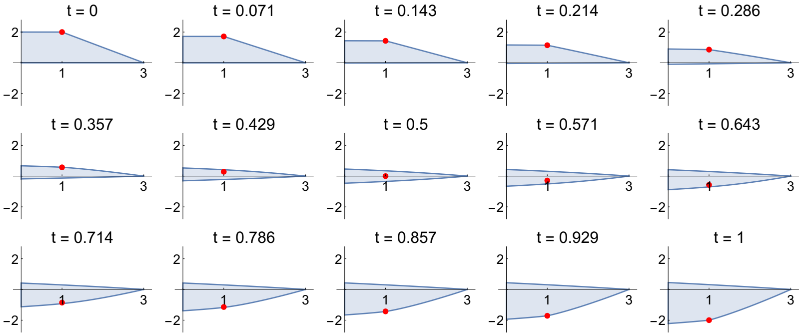
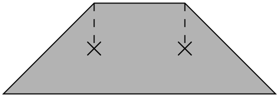


A system with two focus-focus points

- Another semitoric polygon:



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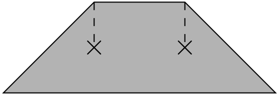
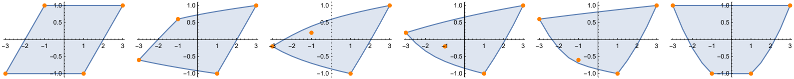


- Think about coupled angular momenta again:



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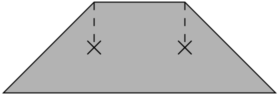

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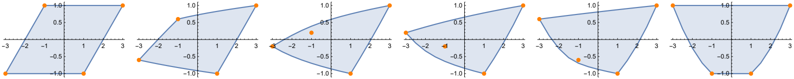

- The point NS passes through the interior and becomes focus-focus, can we do this with SN as well?

The semitoric polygons

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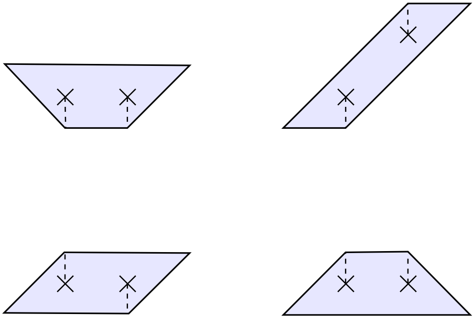

A two parameter family

Let $J = R_1 z_1 + R_2 z_2$ and

$$\begin{cases} H_{0,0} & = x_1 x_2 + y_1 y_2 + z_1 z_2 \\ H_{1,0} & = z_1 \\ H_{0,1} & = z_2 \\ H_{1,1} & = x_1 x_2 + y_1 y_2 - z_1 z_2 \end{cases}$$

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$$H_{s_1,s_2} = (1-s_2)\Big((1-s_1)H_{0,0} + s_1 H_{1,0}\Big) + s_2\Big((1-s_1)H_{0,1} + s_1 H_{1,1}\Big).$$

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Theorem (Hohloch-P., 2018)

Let $R_1 = 1$ and $R_2 = 2$. Then $(J, H_{\frac{1}{2},\frac{1}{2}})$ is a semitoric integrable system with exactly two focus-focus points (and so is every system in an open neighborhood of these parameters).

The momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$

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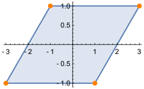

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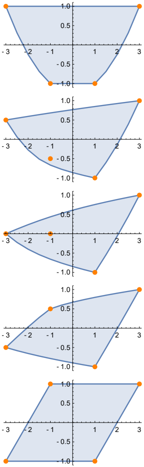

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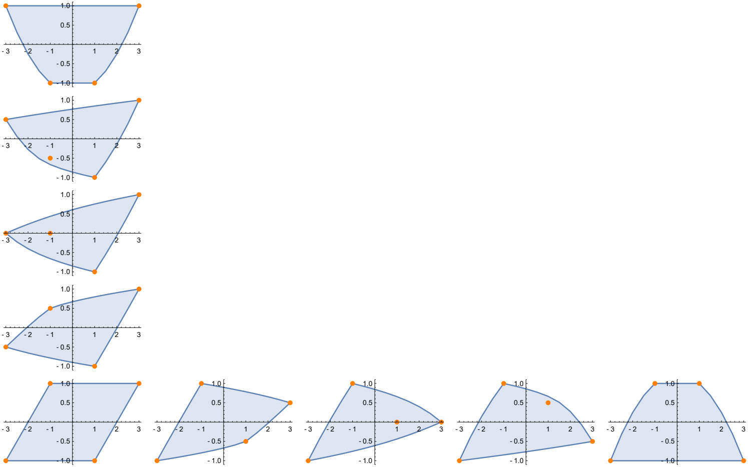

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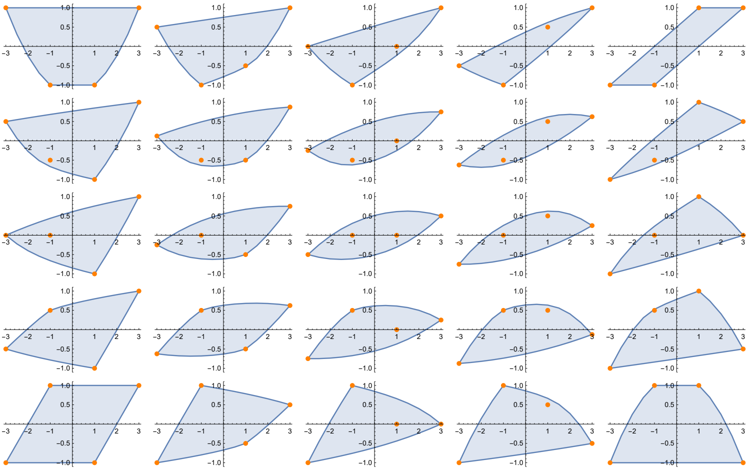

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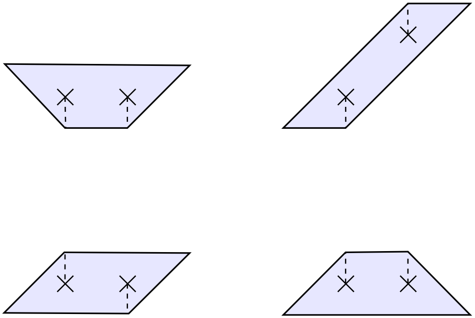

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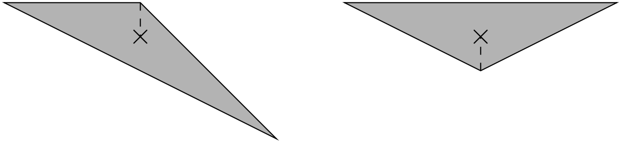

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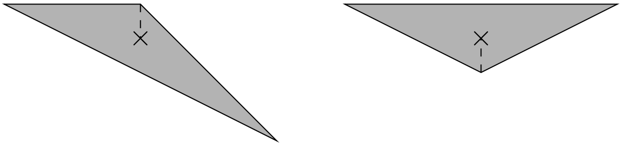

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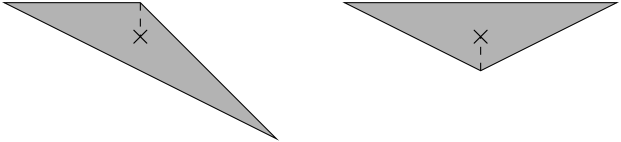

- The right polygon does not correspond to a toric system!
  - This means we cannot use a semitoric transition family in the same way (when the focus-focus point collides with the boundary)

Z_k-spheres

- $F_t = (J, H_t)$ means that the underlying S^1-manifold (M, ω, J) is fixed.

Z_k-spheres

- $F_t = (J, H_t)$ means that the underlying S^1-manifold (M, ω, J) is fixed.
- The relationship between S^1-spaces and semitoric systems was studied by Hohloch-Sabatini-Sepe.

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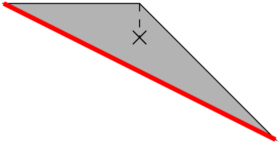

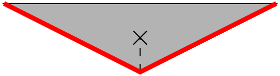

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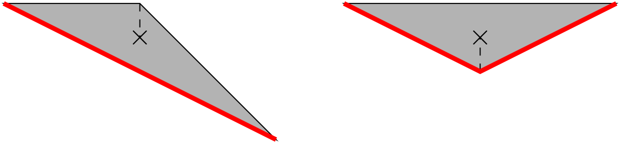

- Points in Z_k-spheres are automatically singular points of the integrable system, but in toric and semitoric systems *lines of singular points cannot enter the interior of $F(M)$.*

A system on $\mathbb{CP}^2$

- λ, δ, γ are parameters satisfying $0 < \gamma < \frac{1}{4\lambda}$ and $\delta > \frac{1}{2\gamma\lambda}$.
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- last term "pushes" the Z_2-sphere to keep it on the boundary.

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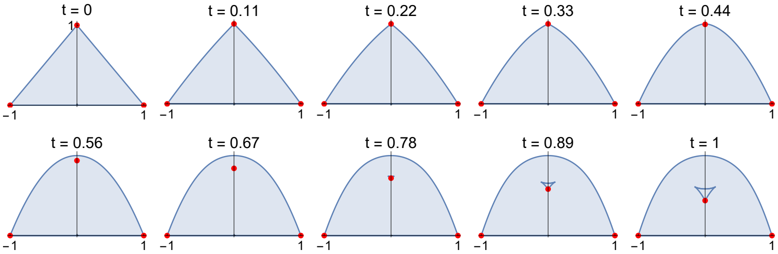

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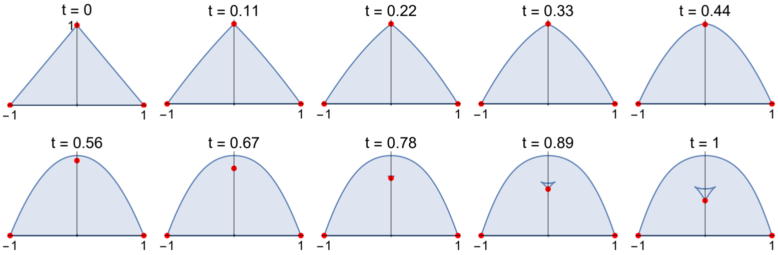

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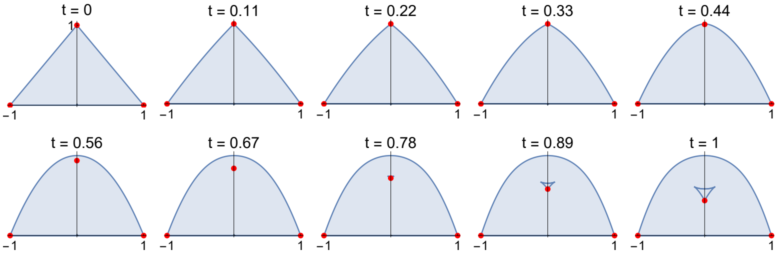

- For large t the system develops a flap, including hyperbolic-regular points and parabolic points
- the transition point still changes EE to FF to EE, but it can't merge with the bottom boundary (the Z_2-sphere) so instead it forms a flap.

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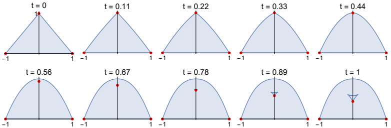

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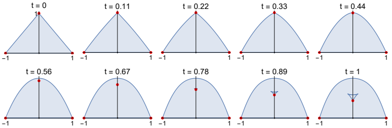

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The family $(\mathbb{CP}^2, n\omega_{FS}, F_t = (J, H_t))_{0 \leq t \leq 1}$ *is*

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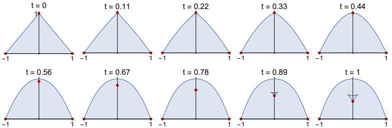

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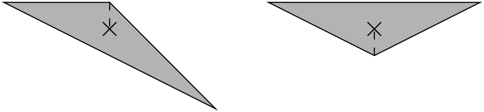
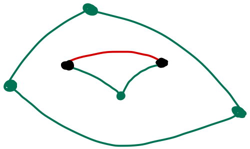


Flaps in integrable systems

Fibers in a "flap"



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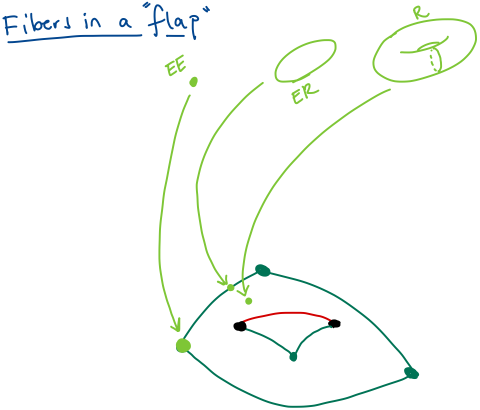

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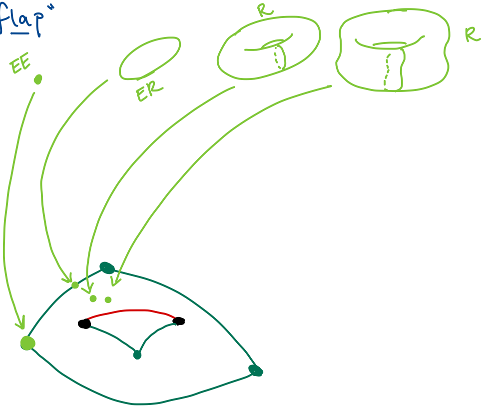

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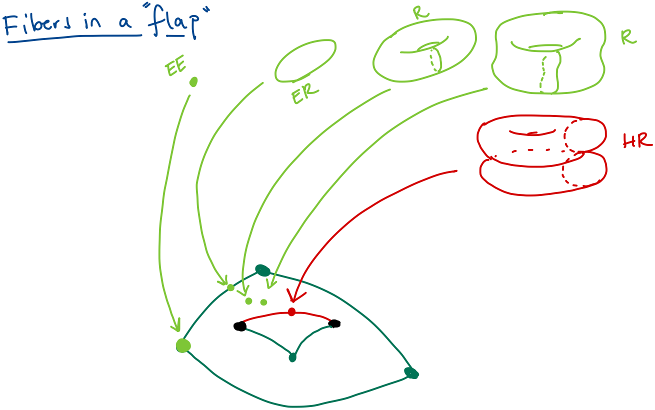

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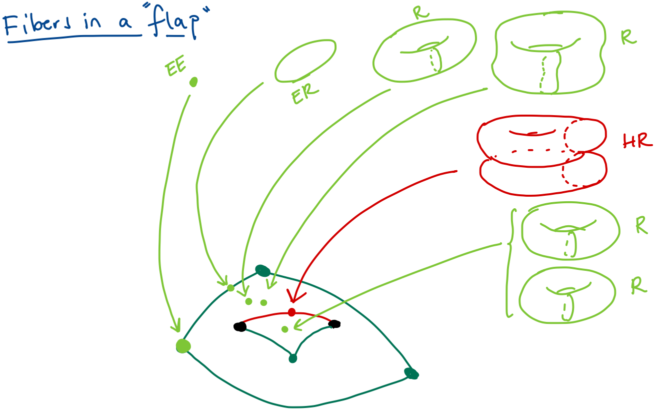

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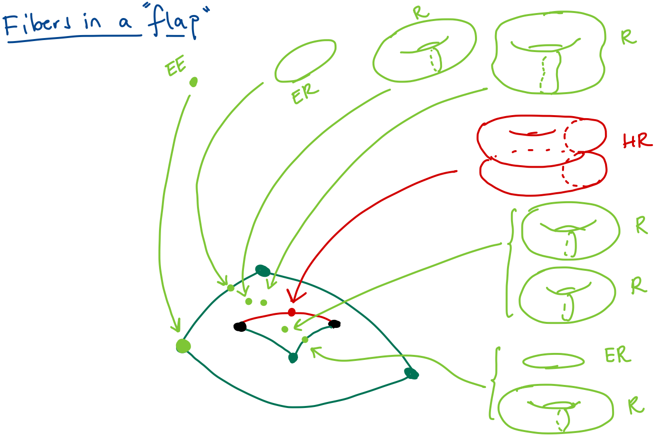

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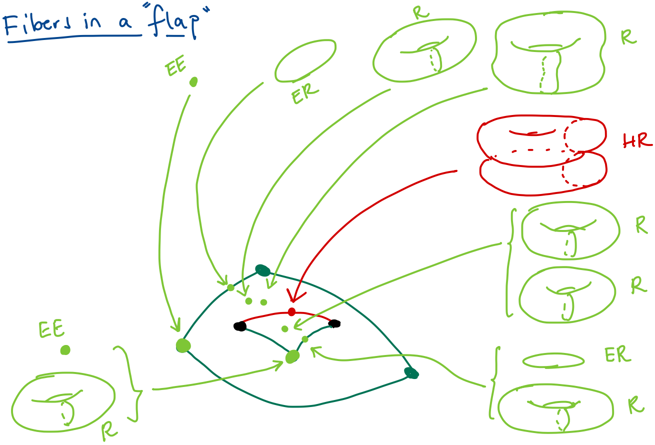

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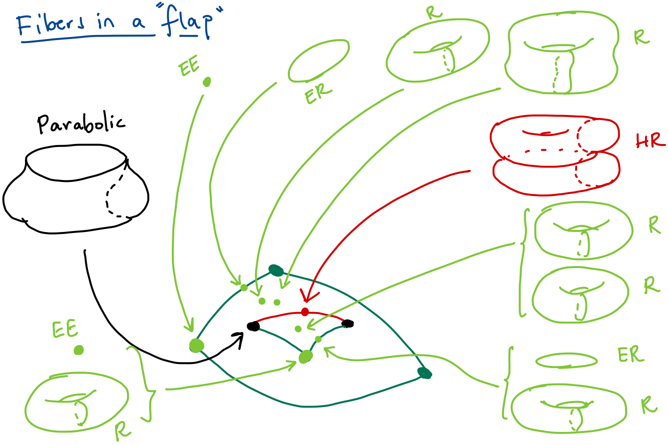

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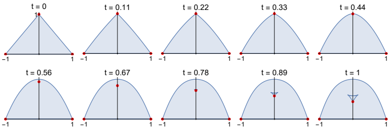
- Study various properties of the fibers (non-displacible?, Hamiltonian isotopic?, heavy or superheavy?, Floer theory?, etc...)

Thanks!



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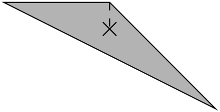

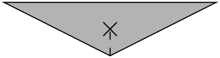

"Appendix"

Some extra slides

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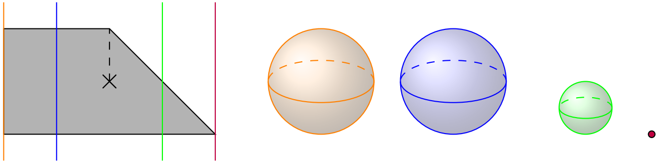

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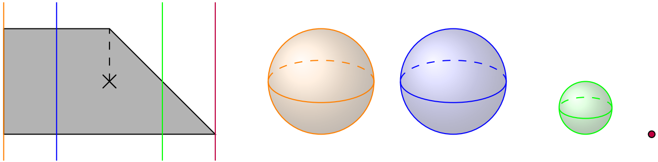
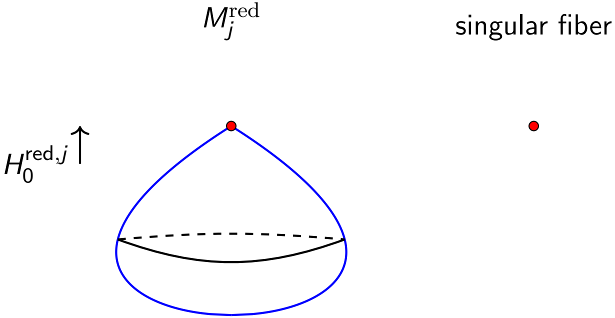


- If $dJ_j = 0$ get a 'teardrop' or 'pinched sphere' singular space.

Coupled angular momentum: reduction



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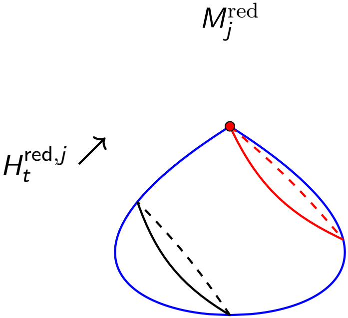
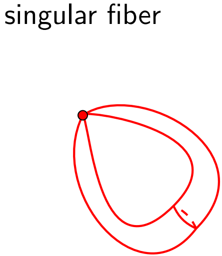
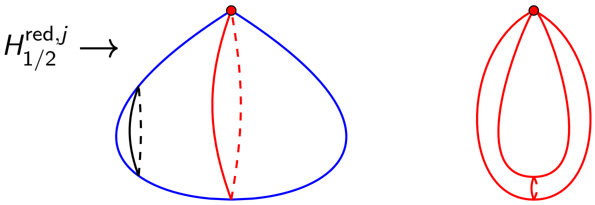




Coupled angular momentum: reduction

M_j^{red} singular fiber



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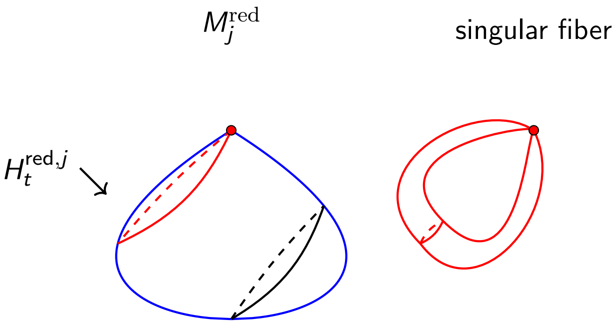

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