Dynamic Inefficiency in Decentralized Capital Markets*

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February 17, 2016

Abstract

We study the efficiency implications of bargaining in frictional capital markets in which firms match bilaterally with dealers in order to buy or sell capital. We show how two of the distinguishing characteristics of capital – ownership and the intertemporal nature of investment – give rise to a dynamic inefficiency. Firms that anticipate buying capital in the future overinvest because this increases their outside option of no trade in negotiations with dealers in the future, thereby lowering the bargained purchase price. Vice versa, firms that anticipate selling capital in the future strategically underinvest because this increases the bargained sale price. If the only motive for trade is capital depreciation, there is overinvestment in capital. With stochastic productivity, there is insufficient dispersion of capital across firms and investment is insufficiently responsive to shocks. A regressive tax on capital can restore the efficient capital allocation.

*We thank Jun Ishii, Ricardo Lagos, Miguel Leon-Ledesma, Guillaume Rocheteau, and participants at the Federal Reserve Board's brown bag seminar, the Search and Matching Workshop at the Federal Reserve Bank of Philadelphia, the 2013 meetings of the Canadian Economic Association, the 2013 China meeting of the Econometric Society, the 2013 Summer Workshop on Money, Banking, Payments and Finance at the Federal Reserve Bank of Chicago, and the 2013 European meeting of the Econometric Society for helpful comments and discussions.

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1 Introduction

Many forms of capital trade in frictional decentralized markets. This is true both for physical capital, such as structures and equipment, and for financial assets, which frequently trade in so-called over-the-counter (OTC) markets.[1] A burgeoning literature, reviewed below, has emerged that uses search and bargaining models to characterize the frictions involved in these markets and to rationalize important phenomena such as trade volume, bid-ask spreads, or trading delays. This departure from the Walrasian paradigm raises a classic question: To what extent do these markets achieve efficiency? In particular, is the level of investment efficient and if not, is there a tax policy that can restore efficiency?

In this paper, we study the efficiency implications of bargaining in a frictional capital market in which production firms match bilaterally with dealers in order to buy or sell capital. Our framework nests both the standard model of investment in physical capital, augmented with trading frictions, and the model of financial asset trade in OTC markets considered by Lagos and Rocheteau (2009). We show how two of the distinguishing characteristics of capital – ownership and the intertemporal nature of investment – provide a dynamic strategic incentive for firms that generally leaves the decentralized equilibrium constrained inefficient. The distortion arises from the fact that firms' current investment decision affects future bargaining outcomes. In particular, firms that anticipate buying capital in the future strategically overinvest because this increases their outside option of no trade in negotiations with dealers in the future, thereby lowering the bargained purchase price. Vice versa, firms that anticipate selling capital in the future strategically underinvest because this increases the bargained sale price.

This insight leads to three key results. First, the presence of capital depreciation always creates an incentive to overinvest. However, the degree of overinvestment is non-monotonic in the depreciation rate because depreciation not only increases the need to buy capital in the future but also reduces the strategic value of current investment for future negotiations.

[1] See Dell'Ariccia and Garibaldi (2005), Duffie, Garleanu and Pedersen (2005, 2007); Green, Li and Schuerhoff (2010); or Afonso and Lagos (2012a) for examples of decentralized financial asset markets that are subject to trading frictions and bargaining. See Rauch (1999) and Nunn (2007) for a list of real asset markets without organized exchange nor reference prices in trade publications. Also see Pulvino (1998) and Gavazza (2011) for the quantitative relevance of trading frictions and bargaining in commercial aircraft markets – presumably one of the most homogenous and frictionless real asset markets

Second, stochastic productivity can lead to either overinvestment or underinvestment. For a mean-reverting productivity process, high-productivity firms underinvest since they expect to be sellers of capital in the future, and low-productivity firms overinvest since they expect to be buyers of capital in the future. As a consequence, the dispersion of capital across firms and trade volume are too low relative to the social optimum, and investment is insufficiently responsive to shocks. These implications provide an interesting contrast to the notion that dispersed capital holdings, high trade volume, and business cycle volatility are symptoms of inefficiency.

Third, a tax on capital can restore the efficient allocation. The tax is regressive, with a positive marginal tax rate for low levels of capital and a negative marginal tax rate for high levels of capital. For the case of zero depreciation – which is the case relevant for the literature on financial asset trade in OTC markets exemplified by Lagos and Rocheteau (2009) – the optimal corrective tax is equivalent to a wealth tax.

The paper is related to different literatures. The paper contributes to the now extensive literature on search theoretical models of financial asset trade in OTC markets, pioneered by Duffie, Garleanu and Pedersen (2005) and extended by Lagos and Rocheteau (2009) to allow for unrestricted asset holdings.[2,3] The focus of this literature has so far been primarily on the positive implications of trading frictions. By contrast, our focus is on analyzing the normative implications. To our knowledge, we are the first to explicitly characterize the inefficiency arising in the Lagos-Rocheteau environment. In particular, we establish that, as a result of the inefficiency, asset holdings tends to be insufficiently responsive to productivity relative to the social optimum; and that a regressive tax on capital – or equivalently a wealth tax for the case of zero depreciation – can restore the efficient allocation.[4]

[2]Other important contributions to this literature include Duffie, Garleanu and Pedersen (2007), Weill (2007), Vayanos and Weill (2008), Afonso and Lagos (2012b), Lester, Rocheteau and Weill (2014), or Hugonnier, Lester and Weill (2015) among many others. Also see Williamson and Wright (2010) for a review of search-theoretic models of trade in OTC markets.

[3]The search-theoretic literature on trading frictions in OTC markets typically considers asset trade between risk-averse investors with stochastic valuation for the asset and risk-neutral dealers. In terms of mechanics, this is equivalent to our setup with firms as long as the production technology is concave in capital. Also see Nosal and Rocheteau (2011) for such a reinterpretation of the Lagos-Rocheteau model.

[4]It is important to contrast the inefficiency highlighted here to the misallocation of resources that occurs in first-generation models of OTC markets, such as Duffie et al. (2005), in which asset holdings are indivisible. In those models, search frictions reduce welfare by preventing immediate reallocation of capital from low-valuation to high-valuation investors. However, the equilibrium in such a setting is usually constrained

The inefficiency analysis of our paper shares the common insight with the literature on overhiring in frictional labor markets initiated by Stole and Zwiebel (1996) and Smith (1999) that firms have a strategic incentive to distort input factor choices so as to improve their bargaining position.[5] To our knowledge, we are the first to point out the relationship between the overhiring result of Stole and Zwiebel (1996) and the inefficiency present in Lagos-Rocheteau type models of OTC markets. The source of the inefficiency is quite different, however. In the Stole-Zwiebel-Smith environment, firms decide on employment and then bargain pairwise with each worker over the wage. Firms have an incentive to overhire so as to increase the outside option in case of breakdown in negotiations, thereby lowering the wage. In our environment, by contrast, the firm bargains with only one dealer per period in a way that is bilaterally efficient. If the firm invested only once or if the environment was static, there would be no distortion, whereas this is not the case in the Stole-Zwiebel-Smith environment.[6] Instead, the distortion in our environment arises because firms purchase capital, making the capital stock a state variable for the firm's outside option in negotiations with dealers in the future. The inefficiency in our environment therefore stems from the intertemporal nature of investment, which is why we call it a *dynamic inefficiency*. Our key results, including the effects of depreciation and of the productivity process, as well as the implications for the dispersion of capital holdings, trade volume, the responsiveness of investment to shocks, and optimal capital taxation, depend critically on this feature.

Lastly, by setting the analysis in an otherwise standard intertemporal investment model with capital depreciation, our insights can be applied to a modern dynamic macroeconomic context. As such, the paper relates to the emerging literature on trading frictions in physical capital markets, exemplified by Kurmann and Petrosky-Nadeau (2009), Gavazza (2011), Kurmann (2014), Shi and Cao (2014), or Ottonello (2015). This literature is considerably

efficient: there is no intensive margin to be distorted, and conditional on matching, any potential mutually beneficial trades take place. In contrast, with an operative intensive margin as in Lagos and Rocheteau (2009), the equilibrium is constrained inefficient, despite the fact that each trade between a firm and a dealer is bilaterally efficient. As a result, a government intervention can improve welfare by taxing or subsidizing capital, even though the government cannot eliminate the search friction.

[5] Also see Cahuc and Wasmer (2001) and Cahuc, Marque and Wasmer (2008), or Elsby and Michaels (2013).

[6] Indeed, the overhiring result in the Stole-Zwiebel-Smith environment would not arise if workers bargained as a single coalition. Also see the discussion in Section 2.4.

smaller than the above-mentioned body of research on OTC financial markets, and has focused, with the exception of Kurmann (2014), on positive implications. Our paper contributes to this literature by showing that in frictional capital markets with bargaining, the assumption of firms purchasing and owning capital has substantially different normative implications than the assumption of firms renting capital on a period-by-period basis, which is the case analyzed by Kurmann (2014).[7] As explained above, the main results of our paper – including the non-monotonic dependence of overinvestment on depreciation, insufficient dispersion of capital holdings and insufficient responsiveness to shocks – hinge on the dynamic nature of the problem and hence do not appear in Kurmann (2014). Moreover, we establish that the dynamic inefficiency implied by capital ownership provides a novel rationale for a regressive tax on capital.

The paper proceeds as follows. In Section 2, we show the dynamic inefficiency result in a simple deterministic version of the model, in which depreciation is the only motive for trade. Section 3 extends the model to incorporate stochastic productivity and studies how the magnitude and direction of the inefficiency depends on both depreciation and the productivity process. Section 4 discuses the implications of the dynamic inefficiency for the dispersion of capital across firms, the responsiveness of investment to shocks, and optimal taxation. Section 5 concludes.

2 Inefficiency in a deterministic model

This section illustrates the dynamic inefficiency result in a deterministic model of investment with a trading friction. The environment is deliberately simple, and the trading friction represents a minimal departure from the standard intertemporal competitive investment problem. Robustness to alternative modeling assumptions is discussed at the end of the section. Most proofs are relegated to the Appendix.

[7] The inefficiency in Kurmann (2014) is, from a mechanical point of view, the same as in Stole and Zwiebel (1996) and Smith (1999). Firms bargain *pairwise* with all suppliers. The resulting overinvestment incentive stems from the inability to contract efficiently within the period (e.g. suppliers cannot form a coalition and bargain over both rental rate and capital simultaneously).

2.1 Environment

Time is discrete and discounted at rate β. The economy is populated by a continuum of two types of agents: *firms* and *dealers*. In the context of a physical capital market, dealers should be interpreted as suppliers of capital, but we refer to them as dealers throughout the paper, because our analysis also encompasses OTC financial markets.

There is a single consumption good, and both firms and dealers derive linear utility from consumption. Dealers have a linear technology for converting consumption goods into capital and vice versa. Without loss of generality, we assume that the marginal cost of capital in units of consumption is one.[8] Firms produce the consumption good with capital k using technology $f(k)$, where f is a continuous, twice differentiable, strictly increasing and strictly concave function with $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Capital is perfectly divisible and depreciates at rate $\delta \in [0, 1]$.

The allocation of new capital from dealers to firms is subject to trading frictions. In each period, a firm has probability $\lambda \in (0, 1]$ to randomly match with a dealer. In case there is a match, the firm and the dealer Nash bargain over the price of capital ρ for the quantity of new capital x that the firm demands.[9] If negotiation is successful, the dealer delivers x units of capital in exchange for ρx units of consumption, and the firm continues into the next period with capital stock

$$k' = (1 - \delta) k + x. \tag{1}$$

In case there is no match, which occurs with probability $(1 - \lambda)$, or if negotiations are unsucessful, the firm continues into the next period with capital stock $(1 - \delta)k$.

[8] In the neoclassical growth model, the marginal cost of consumption goods for dealers (i.e. capital suppliers) is the real interest rate, which depends on the discount rate and consumer preferences. Alternatively, in Lagos and Rocheteau (2009), the tradable asset is available in fixed supply and there is a frictionless competitive inter-dealer market that determines the marginal cost. In both cases, dealers take the marginal cost as exogenous and so, normalizing it to one does not matter for our results.

[9] An alternative but equivalent formulation is one in which the firm and the dealer Nash bargain simultaneously over both price and quantity. We prefer the present formulation because it affords a direct comparison to the standard intertemporal investment problem.

2.2 Efficient investment

Consider first the problem of a social planner who chooses the optimal time path of capital holdings for a firm that enters the period with capital stock k. The planner's value for this firm can be described recursively as

$$\Upsilon(k) = f(k) + \lambda \max_{k'} \left[-\left(k' - (1-\delta)k\right) + \beta\Upsilon(k')\right] + (1-\lambda)\beta\Upsilon\left((1-\delta)k\right), \tag{2}$$

where k' is restricted to lie in $\left[0, \overline{k}\right]$, with the upper bound $\overline{k}$ large enough that it will not bind at the optimum. We guess – and verify below – that the planner's optimal choice of k' is independent of current k. Then, by iterating forward, the functional equation (2) can be expressed as

$$\Upsilon(k) = \Upsilon(0) + \sum_{t=0}^{\infty} (\beta(1-\lambda))^t \left[f\left((1-\delta)^t k\right) + \lambda(1-\delta)^{t+1} k\right], \tag{3}$$

where

$$\Upsilon(0) = \lambda \max_{k'} (-k' + \beta\Upsilon(k')) + (1-\lambda)\beta\Upsilon(0) \tag{4}$$

is the planner's value of a firm with zero capital. The expression for $\Upsilon(0)$ obtains because the optimal choice of k' does not depend on k, no matter the number of periods since the last match. Hence, $\sum_{t=0}^{\infty}(\beta(1-\lambda))^t[\lambda \max_{k'}(-k' + \beta\Upsilon(k'))] = \frac{\lambda \max_{k'}(-k'+\beta\Upsilon(k'))}{1-\beta(1-\lambda)}$, which is equivalent to (4).

Given (3) and (4), the planner's problem can be characterized as follows.

Proposition 1. $\Upsilon(k)$ *is continuous, twice differentiable, strictly increasing and strictly concave. There exists a unique efficient level of capital* k^P*, determined by the first-order condition*

$$1 = \beta \sum_{t=0}^{\infty} (\beta(1-\lambda)(1-\delta))^t \left[f'\left((1-\delta)^t k^P\right) + \lambda(1-\delta)\right] \tag{5}$$

Proof. See Appendix. □

Note that for $\lambda = 1$, the problem reduces to the standard intertemporal investment problem, whose solution is described by the well-known Euler equation $1 = \beta\left[f'(k') + (1-\delta)\right]$.

The presence of additional terms in (5) reflects the planner's concern that the firm does not match with a dealer in the future, which occurs with probability $(1-\lambda)$.

For the efficiency analysis conducted below, it is useful to examine the comparative statics properties of k^P with respect to the degree of trading frictions, λ.

Lemma 2. *Let k^P be implicitly defined as a function of λ through equation (5). Then, $\partial k^P/\partial\lambda \leq 0$, with strict inequality as long as $\delta \in (0,1)$.*

Proof. See Appendix. □

Intuitively, the trading friction gives rise to a "precautionary investment motive". When λ is small, there is a high probability that the firm will be unable to purchase capital in the future, which means that there will be periods when the firm's marginal product of capital is higher than the marginal cost. Due to the concavity of the production function, the planner has the firm insure against this possibility by accumulating a higher capital stock today, thereby reducing the firm's trading needs in the future.[10] The higher λ, the smaller this precautionary investment motive and therefore the smaller the efficient level of capital.

2.3 Inefficient investment in the decentralized economy

Consider now the decentralized equilibrium allocation. The value of a firm that enters the period with capital k can be described as

$$v(k) = f(k) + \lambda \max_{k'} \left[-\rho(k,k')(k' - (1-\delta)k) + \beta v(k')\right] + (1-\lambda)\beta v((1-\delta)k), \qquad (6)$$

where the price of capital $\rho(k,k')$ solves the generalized Nash bargaining problem

$$\rho(k,k') = \arg\max_{\rho} S_d^{\phi} S_f^{1-\phi}, \qquad (7)$$

with S_d and S_f representing the dealer's and the firm's surplus, respectively, from trading $x = k' - (1-\delta)k$ units of new capital at price ρ; and $\phi \in [0,1]$ denoting the dealer's

[10] This precautionary investment motive is equivalent to the "liquidity hedging" motive, highlighted by Lagos and Rocheteau (2008) in OTC markets with unrestricted asset holdings.

bargaining weight. The dealer's surplus from trade at price ρ is

$$S_d = (\rho - 1)\left(k' - (1-\delta)\,k\right). \tag{8}$$

The firm's surplus from trade is

$$S_f = -\rho\left(k' - (1-\delta)\,k\right) + \beta v\left(k'\right) - \beta v\left((1-\delta)\,k\right). \tag{9}$$

Substituting (8) and (9) into (7) and solving the bargaining problem yields

$$\rho\left(k, k'\right) = 1 + \phi\left[\frac{\beta v\left(k'\right) - \beta v\left((1-\delta)\,k\right)}{k' - (1-\delta)\,k} - 1\right]. \tag{10}$$

The bargained price of capital equals the dealer's marginal cost of producing capital, which equals unity, plus the dealer's share ϕ of the match surplus per unit of investment, which is the difference between the firm's average gain from continuing with k' as opposed to $(1-\delta)k$ and the supplier's marginal cost. The expression confirms that, as stated in (6), the price of capital is a function of the firm's capital stock k with which it enters the period and the capital stock k' it would like to have in the next period.

Equation (10) can be used to substitute out $\rho\left(k, k'\right)$ in (6) to obtain

$$v\left(k\right) = f\left(k\right) + \lambda(1-\phi)(1-\delta)k + \lambda(1-\phi)\max_{k'}\left[-k' + \beta v\left(k'\right)\right] + (1-\lambda(1-\phi))\beta v((1-\delta)k). \tag{11}$$

Comparison of (11) with (2) shows that the firm's problem is of the same form as the planner's problem, with the only difference that the match probability λ is replaced by $\lambda(1-\phi)$. Hence, the firm's optimal k' is independent of current k, and the functional equation (11) can be expressed in iterated form as

$$v\left(k\right) = v\left(0\right) + \sum_{t=0}^{\infty}\left(\beta(1-\lambda\left(1-\phi\right))\right)^{t}\left[f\left((1-\delta)^{t} k\right) + \lambda\left(1-\phi\right)\left(1-\delta\right)^{t+1} k\right], \tag{12}$$

where

$$v\left(0\right) = \lambda(1-\phi)\max_{k'}(-k' + \beta v\left(k'\right)) + \left(1-\lambda(1-\phi)\right)\beta v\left(0\right). \tag{13}$$

By Proposition 1, the firm's value $v(k)$ has the same properties as the planner's value $\Upsilon(k)$, and there exists a unique decentralized level of capital k^D, determined by the first-order condition

$$1 = \beta \sum_{t=0}^{\infty} (\beta(1 - \lambda(1 - \phi))(1 - \delta))^t \left[f'\left((1 - \delta)^t k^D\right) + \lambda(1 - \phi)(1 - \delta) \right]. \tag{14}$$

Comparison of (14) with (5) then yields the following overinvestment result.

Proposition 3. *$k^D \geq k^P$ with strict inequality as long as $\delta \in (0, 1)$ and $\phi \in (0, 1]$.*

Proof. Since $\lambda(1 - \phi) < \lambda$, the result follows directly from Lemma 2. □

To understand this overinvestment result, note that the firm's problem is payoff equivalent to the one in an alternative environment in which the firm had all the bargaining power but the probability of meeting with a dealer was $\lambda(1 - \phi) < \lambda$.[11] Both the social planner and the firm in the decentralized economy care about the possibility of not trading in a particular period, which gives rise to the precautionary investment motive discussed above. The firm, however, also cares about the value of its capital in the off-equilibrium case when negotiations with the dealer break down, because this outside option affects the terms of trade. This "strategic investment motive" effectively reduces the rate at which the firm discounts future payoffs in the event of no trade (i.e. $\beta(1 - \lambda(1 - \phi)) > \beta(1 - \lambda)$).

Intuition for the overinvestment result can also be gained by inspecting the price equation in (10). First, notice that by (14) and the strict concavity of $v(k)$, $\frac{\beta v(k') - \beta v((1-\delta)k)}{k' - (1-\delta)k} > \beta v'(k') = 1$. Hence $\rho(k', k) > 1$ for all k' and k as long as $\delta \in (0, 1)$ and $\phi \in (0, 1]$; i.e. the dealer extracts rents above its marginal cost from the firm. Further, by the strict concavity of $v(k)$, $\rho(k', k)$ is decreasing in k. All else equal, the firm therefore has a dynamic incentive to overinvest today because this decreases the rents extracted by dealers in future trades.

To round out this intuition, consider the limiting cases of $\delta = 0$ (no depreciation) and $\delta = 1$ (full depreciation).

Corollary 4. *For $\delta = 0$ or $\delta = 1$, the equilibrium is efficient: $k^D = k^P$.*

[11] This equivalence is the same as the one noted by Lagos and Rocheteau (2009) in their OTC environment. The difference is that they do not exploit this equivalence to analyze the efficiency consequences of their trading friction.

Proof. See Appendix. □

For $\delta = 0$, the firm purchases capital only once, and there is no future motive for trade. Hence, the dynamic incentive to overinvest falls away because, at the time it purchases its capital, the firm is not concerned about bargaining with future dealers. For $\delta = 1$, the firm's entire capital stock is exhausted in each period and therefore does not affect the firm's outside option in future trades. Hence, the dynamic incentive to overinvest also falls away. Unlike for the case of $\delta = 0$, this is not because the firm does not need to bargain with dealers in the future, but because its current choice is irrelevant for its future bargaining position. The two limiting cases illustrate that the degree of overinvestment is non-monotonic in δ: a higher δ increases the motive for trade in the future, but reduces the strategic value of current investment for future negotations.

2.4 Discussion of modeling choices

The model considered here is deliberately simple so as to illustrate the forces that drive the dynamic inefficiency result. The trading friction is the only departure from the standard neoclassical intertemporal investment problem. This trading friction generates a match surplus, which opens the door to bargaining and strategic behavior. The resulting distortion is robust to various modifications of the environment. In particular, the dynamic inefficiency result obtains if the dealer's cost of producing new capital is convex instead of linear; if the firm's investment decision is subject to adjustment costs; if the firm has a choice of capital utilization and labor input; or if the firm can vary its search effort for a supplier and as a result, the match probability is endogenous.[12]

Also note that the inefficiency result does not require $\lambda < 1$ (infrequent matching). It only requires that in the event of a breakdown in negotiations, trading frictions prevent the firm from finding another dealer within the same period. Specifically, for $\lambda = 1$ (matching in every period), the firm's capital stock k^D is determined by $1 = \beta \sum_{t=0}^{\infty} (\beta\phi(1 - \delta))^t \left[f'\left((1-\delta)^t k'\right) + (1-\phi)(1-\delta)\right]$ whereas the efficient capital stock k^P is determined

[12] Results for these extensions are available by request. If the production cost of capital for dealers or the investment adjustment cost for firms is convex, proving existence of a unique equilibrium and deriving its properties becomes more involved because the firm's optimal new capital choice is no longer independent of its current capital stock.

by the Euler equation $1 = \beta [f'(k') + (1-\delta)]$. The difference between the two first-order conditions highlights one more time the crucial difference between the planner's and the firm's objective. The planner only cares about equalizing the marginal product of capital to the opportunity cost of capital. The firm, by contrast, also cares about the strategic value of its capital in negotiations with future dealers.

The trading friction is specified so as not to give rise to a standard holdup problem, as considered for example by Grout (1984), Caballero and Hammour (1998), or Acemoglu and Shimer (1999).[13] In particular, the bargaining outcome between the firm and the dealer in the model is bilaterally Pareto efficient; i.e. the level of capital k^D is what would be chosen if the firm and the dealer bargained simultaneously over both price and quantity, and hence no other level of capital can simultaneously improve both the firm's and the dealer's situation. As a result, the inefficiency would be absent in a one-shot version of the model, as illustrated by the above limiting cases of no depreciation (the firm only invests once) and full depreciation (current capital does not matter for future negotiations).

As discussed in the introduction, the dynamic nature of the inefficiency constitutes an important difference to the literature on overhiring in frictional labor markets initiated by Stole and Zwiebel (1996) and Smith (1999). In their environment, the firm sets employment and then bargains pairwise with all workers. Overhiring arises because the firm has a strategic incentive *within* the period to lower its outside option in case of breakdown in negotiations with any one of the workers, independent of whether the environment is dynamic or not. Similarly, in Kurmann (2013), overinvestment arises because firms rent capital and bargain pairwise with each dealer on a period-by-period basis. The rental assumption effectively removes the dynamic inefficiency because the firm's capital stock is no longer a state variable in negotiations with dealers. This comparison highlights that it is the assumption of ownership of capital by the firm in conjunction with the intertemporal nature of capital accumulation that gives rise to the dynamic inefficiency.

[13] In particular, neither the firm nor the dealer are subject to ex-ante fixed costs that would be open to ex-post appropriation by the other party.

3 Inefficiency in a stochastic model

In the deterministic model considered above, depreciation was the only motive for trade, and the firm was always a buyer of capital. We now extend the model to allow for stochastic shocks to productivity, which generate a second motive for trade. The resulting framework nests both the model in Section 2 and a discrete-time analogue of the OTC asset market considered in Lagos and Rocheteau (2009) and Nosal and Rocheteau (2011), which has stochastic asset valuation but no depreciation.

We show that the extension to a stochastic environment does not change the logic behind the dynamic inefficiency that we have identified in Section 2. However, shocks to productivity imply that firms have an incentive to either buy or sell capital, depending on the size of the shock and the properties of the productivity process; i.e. the direction of the inefficiency becomes ambiguous. A firm overinvests, as before, if it anticipates being a buyer of capital in the future, but underinvests if it anticipates being a seller in the future. For a mean-reverting productivity process, we characterize the productivity threshold below which the firm overinvests and show that this threshold depends on a simple tradeoff between the persistence of productivity and the magnitude of capital depreciation.

3.1 Environment

We modify the framework from Section 2 by assuming that the production technology takes the form $zf(k)$, where z denotes productivity.[14] The timing is as follows. A firm enters the period with a capital level k, draws productivity z, and produces output $zf(k)$. It then meets a dealer with probability λ. In case of a meeting, the firm chooses next period capital level k', which may be larger or smaller than $(1-\delta)k$. In the absence of a meeting, the firm proceeds to the next period with capital stock $(1-\delta)k$.

We assume that z follows a Markov process with a transition probability $\pi(z'|z)$ satisfying $\sum_{z'} \pi(z'|z) = 1 \; \forall z$. For much of our analysis, we will be interested in the expectation of t period ahead productivity, z_t, conditional on current productivity realization $z_0 = z$, defined

[14] For the purpose of our analysis, it is not important whether productivity is an aggregate or an idiosyncratic state. This is because the bargaining problem between a firm and a dealer depends only on the firm's productivity.

as

$$\xi_t(z) \equiv \mathbb{E}(z_t|z_0 = z) = \sum_{z_1} \pi(z_1|z_0) \sum_{z_2} \pi(z_2|z_1) \ldots \sum_{z_t} \pi(z_t|z_{t-1}) z_t. \tag{15}$$

3.2 Efficient investment

Consider a firm that enters the period with capital k and productivity z. Analogous to the deterministic environment of the previous section, the planner's problem for this firm can be described recursively as

$$\begin{aligned} \Upsilon(k,z) &= zf(k) + \lambda \max_{k'} \left(\beta \sum_{z'} \pi(z'|z) \Upsilon(k', z') - (k' - (1-\delta)k) \right) \\ &\quad + (1-\lambda)\beta \sum_{z'} \pi(z'|z) \Upsilon((1-\delta)k, z'), \end{aligned} \tag{16}$$

Guessing – and verifying later – as in Section 2 that the optimal choice of k' is independent of current k, (16) can be iterated forward and expressed as

$$\Upsilon(k,z) = \Upsilon(0,z) + \sum_{t=0}^{\infty} (\beta(1-\lambda))^t \left(\xi_t(z) f((1-\delta)^t k) + \lambda(1-\delta)^{t+1} k \right) \tag{17}$$

where

$$\Upsilon(0,z) = \lambda \sum_{z'} \pi(z'|z) \left(\max_{k'} \beta \Upsilon(k', z') - k' \right) + (1-\lambda)\beta \sum_{z'} \pi(z'|z) \Upsilon(0, z') \tag{18}$$

is the planner's value of a firm with zero capital. Similar to the deterministic problem, this expression for $\Upsilon(0,z)$ obtains because the optimal choice of k' does not depend on k, no matter the number of periods since the last match.

By the properties of f, it is clear that $\Upsilon(k,z)$ is continuous, differentiable and strictly concave in k. Hence, for each shock realization z, there exists a unique efficient level of capital $k'(z) = k^P(z)$, determined by the first-order condition

$$1 = \beta \sum_{z'} \pi(z'|z) \sum_{t=0}^{\infty} (\beta(1-\lambda)(1-\delta))^t \left(\xi_t(z') f'((1-\delta)^t k^P(z)) + \lambda(1-\delta) \right). \tag{19}$$

Condition (19) generalizes Proposition 1 for the case with stochastic productivity. The key difference to the deterministic case is that, while the analogue of Proposition 1 generalizes to the stochastic version, the comparative statics result of Lemma 2 does not; i.e. we do not necessarily have $\partial k^P(z)/\partial\lambda \leq 0$ for all z.

To understand why Lemma 2 does not apply, notice that in the deterministic model, the planner always anticipates that the firm will be a buyer of capital in the future. When λ is small, the precautionary investment motive of insuring against future periods of no trade is large and as a result, the planner has the firm accumulate a larger capital stock in the event of trade today. With stochastic productivity shocks, by contrast, the planner may, as a result of an expected decline in productivity, anticipate that the firm will be a seller of capital in the future. In this case, the logic is reversed: the smaller λ, the smaller the capital stock the planner has the firm accumulate in the event of trade for fear of being unable to resell the capital in the future. This intuition is formalized by the following Lemma, which generalizes Lemma 2.

Lemma 5. *For $\delta < 1$, a sufficient condition for $\partial k^P(z)/\partial\lambda < 0$ is*

$$\frac{\xi_t(z)}{\xi_{t+1}(z)} < \frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)} \quad \forall t \geq 1, \forall k$$

Similarly, a sufficient condition for $\partial k^P(z)/\partial\lambda > 0$ is

$$\frac{\xi_t(z)}{\xi_{t+1}(z)} > \frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)} \quad \forall t \geq 1, \forall k$$

Proof. See Appendix. □

As long as the marginal product of capital in the event of no trade in the future is expected to increase over time, the precautionary investment motive and therefore $k^P(z)$ is increasing in the trading friction. Vice versa, if the marginal product of capital in the event of no trade is expected to decline over time, the precautionary investment motive and therefore $k^P(z)$ is decreasing in the trading friction.

For permanent productivity shocks, i.e. when z is expected to remain constant, Lemma 5 reduces to the deterministic model because in this instance, depreciation causes the marginal product of capital in the event of no trade to monotonically increase over time; i.e. the left-hand side in the above conditions is always 1, which is less than $\frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)}$ by the strict concavity of f. So, we recover $\partial k^P(z)/\partial\lambda < 0$ as in Lemma 2.

3.3 Inefficient investment in the decentralized economy

Consider now the decentralized equilibrium allocation; i.e. the stochastic analogue of section 2.3. The value of a firm entering the period with capital stock k and productivity z is

$$\begin{aligned} v(k,z) &= zf(k) + \lambda \max_{k'} \left(\beta \sum_{z'} \pi(z'|z) v(k',z') - \rho(k',k,z)(k'-(1-\delta)k) \right) \\ &\quad + (1-\lambda)\beta \sum_{z'} \pi(z'|z) v((1-\delta)k, z'), \end{aligned} \tag{20}$$

subject to the Nash-bargained solution for the price

$$\rho(k,k',z) = 1 + \phi \left[\sum_{z'} \pi(z'|z) \frac{\beta v(k',z') - \beta v((1-\delta)k, z')}{k' - (1-\delta)k} - 1 \right], \tag{21}$$

which is derived analogously to the deterministic model. Using (21) to substitute out $\rho(k,k',z)$ in (20) yields

$$\begin{aligned} v(k,z) &= zf(k) + \lambda(1-\phi)(1-\delta)k + \lambda(1-\phi) \max_{k'} \left(\beta \sum_{z'} \pi(z'|z) v(k',z') - k' \right) \\ &\quad + (1-\lambda(1-\phi))\beta \sum_{z'} \pi(z'|z) v((1-\delta)k, z'). \end{aligned} \tag{22}$$

Similar to the deterministic case, the firm's problem in (22) is of the same functional form as the planner's problem in (16), with the only difference that λ is replaced by $\lambda(1-\phi)$. Hence, $v(k,z)$ has the same properties as $\Upsilon(0,z)$, and for each shock realization z, there exists a unique optimal decentralized level of capital $k'(z) = k^D(z)$, determined by first-order condition (19) but with λ replaced by $\lambda(1-\phi)$. Using Lemma 5, the investment inefficiency of the decentralized allocation can therefore be characterized as follows.

Proposition 6. *For $\delta < 1$, a sufficient condition for $k^D(z) > k^P(z)$ is*

$$\frac{\xi_t(z)}{\xi_{t+1}(z)} < \frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)} \ \forall t \geq 1, \forall k$$

Similarly, a sufficient condition for $k^D(z) < k^P(z)$ is

$$\frac{\xi_t(z)}{\xi_{t+1}(z)} > \frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)} \ \forall t \geq 1, \forall k$$

Proof. This is a direct consequence of Lemma 5. □

Proposition 6 shows that whether a firm overinvests or underinvests depends on a simple tradeoff between the stochastic process for productivity and the depreciation rate of capital. The intuition for this inefficiency is very similar to the deterministic case. In contrast to the planner, the firm cares not only about the possibility of not trading in the future, but also about the effect its capital stock has on the outside option when bargaining with future dealers. This strategic investment motive effectively reduces the rate at which the firm discounts future payoffs in the event of no trade. If the firm anticipates buying additional capital in the future, which occurs if its marginal product of capital in the event of no trade is expected to rise, the strategic investment motive increases the firm's precautionary investment motive and thus $k^D(z) > k^P(z)$. Vice versa, if the firm anticipates selling capital in the future, which occurs if its marginal product of capital in the event of no trade is expected to decrease, the strategic investment motive is reduced and thus $k^D(z) < k^P(z)$.[15]

To more explicitly characterize which firms overinvest and which firms underinvest requires making specific assumptions about the stochastic process for productivity. Throughout the rest of the analysis, we assume that the stochastic process is such that the conditional expectation $\xi_t(z)$ takes the form

$$\xi_t(z) = \gamma^t z + \left(1 - \gamma^t\right) \overline{z}, \tag{23}$$

[15] It is worthwhile to note that for $\delta = 1$ efficiency obtains, as in Corollary 4; i.e. $k^D(z) = k^P(z)$. Intuitively, when capital fully depreciates every period, it is no longer a state variable for the firm and does not affect its outside option in future negotiations with dealers.

where $\gamma \in [0,1]$ and $\overline{z} > 0$ are constants. This form of conditional expectations arises, for example, from a process in which each firm receives a productivity shock with Poisson arrival rate $1-\gamma$ that is independent across firms. Conditional on the shock, firms draw a new productivity level from a time-invariant distribution Π. Thus, $\pi(z'|z) = \gamma + (1-\gamma)\Pi(z)$ for $z' = z$, and $(1-\gamma)\Pi(z')$ otherwise. In this case, $\overline{z}$ in (23) is the unconditional mean of z; i.e. $z = \sum_j \Pi_j z_j$. Exactly this stochastic process is assumed in Lagos and Rocheteau (2009) and in the literature following it. More generally, notice that any stochastic process generating conditional expectations of the form (23) yields the results below.[16]

Given (23), the optimal choice of capital, and hence necessary and sufficient conditions for overinvestment, can be characterized in closed form for two cases. For $\delta = 0$ – which, notably, encompasses the literature on OTC markets following Lagos and Rocheteau (2009) – Proposition 7 fully characterizes the inefficiency. For general δ, Proposition 8 characterizes the inefficiency under the assumption that the production function takes the power form $f(k) = k^{\alpha}$.

Proposition 7. *Suppose $\delta = 0$.*

1. *$k^P(z)$ solves*

$$\frac{1}{f'(k)} = \frac{\beta}{1-\beta}\left(\tilde{\gamma}^P z + \left(1-\tilde{\gamma}^P\right)\overline{z}\right), \text{ where } \tilde{\gamma}^P = \frac{\gamma - \gamma\beta(1-\lambda)}{1-\gamma\beta(1-\lambda)}. \tag{24}$$

   Similarly, $k^D(z)$ solves

$$\frac{1}{f'(k)} = \frac{\beta}{1-\beta}\left(\tilde{\gamma}^D z + \left(1-\tilde{\gamma}^D\right)\overline{z}\right), \text{ where } \tilde{\gamma}^D = \frac{\gamma - \gamma\beta(1-\lambda(1-\phi))}{1-\gamma\beta(1-\lambda(1-\phi))}. \tag{25}$$

2. *$k^D(z) > k^P(z)$ for $z < \overline{z}$; $k^D(\overline{z}) = k^P(\overline{z})$ for $z = \overline{z}$; and $k^D(z) < k^P(z)$ for $z > \overline{z}$.*

Proof. See Appendix. □

For $\delta = 0$, productivity shocks are the only reason to buy or sell capital. Since, as explained above, the firm cares about the effect its capital stock has on the outside option

[16] For example, the same conditional expectation as in (23) arises if z follows an AR(1) process with $z' = (1-\gamma)\overline{z} + \gamma z + \epsilon$, with ϵ i.i.d.

when bargaining with future dealers, it effectively discounts future payoffs in the event of no trade at a lower rate than the planner. As a result, the firm puts too low of a weight on current productivity relative to future expected productivity, which is reflected by $\tilde{\gamma}^D < \tilde{\gamma}^P$. So, whenever $z < \overline{z}$, the firm overinvests and whenever $z > \overline{z}$, the firm underinvests. Intuitively, firms with below-average z expect their productivity to grow, and hence anticipate buying capital in the future. By increasing their current capital level, they improve their outside option in future capital purchases. Conversely, firms with above-average z expect their productivity to fall, and hence anticipate selling capital in the future. By reducing their capital holdings today, they improve their terms of trade in future capital sales.

Proposition 8. *Suppose $\delta \geq 0$, and the production function has the form $f(k) = k^\alpha$, $\alpha \in (0,1)$.*

1. *$k^P(z)$ solves*
$$k^{1-\alpha} = \frac{\alpha\beta}{1-\beta(1-\delta)}\omega^P\left(\tilde{\gamma}^P z + \left(1-\tilde{\gamma}^P\right)\overline{z}\right), \tag{26}$$
where
$$\omega^P = \frac{1-\beta(1-\lambda)(1-\delta)}{1-\beta(1-\lambda)(1-\delta)^\alpha}, \quad \tilde{\gamma}^P = \frac{\gamma - \gamma\beta(1-\lambda)(1-\delta)^\alpha}{1-\gamma\beta(1-\lambda)(1-\delta)^\alpha} \tag{27}$$
Similarly, $k^D(z)$ solves
$$k^{1-\alpha} = \frac{\alpha\beta}{1-\beta(1-\delta)}\omega^D\left(\tilde{\gamma}^D z + \left(1-\tilde{\gamma}^D\right)\overline{z}\right), \tag{28}$$
where
$$\omega^D = \frac{1-\beta(1-\lambda(1-\phi))(1-\delta)}{1-\beta(1-\lambda(1-\phi))(1-\delta)^\alpha}, \quad \tilde{\gamma}^D = \frac{\gamma - \gamma\beta(1-\lambda(1-\phi))(1-\delta)^\alpha}{1-\gamma\beta(1-\lambda(1-\phi))(1-\delta)^\alpha} \tag{29}$$

2. *If $\gamma(1-\delta)^{\alpha-1} > 1$, then $k^D(z) > k^P(z)$ for all z. If $\gamma(1-\delta)^{\alpha-1} \leq 1$, then $k^D(z) > k^P(z)$ for $z < \hat{z}$, $k^D(\hat{z}) = k^P(\hat{z})$, and $k^D(z) < k^P(z)$ for $z > \hat{z}$, where the threshold $\hat{z}$ is defined by*
$$\hat{z} = \frac{\omega^D\left(1-\tilde{\gamma}^D\right) - \omega^P\left(1-\tilde{\gamma}^P\right)}{\omega^P\tilde{\gamma}^P - \omega^D\tilde{\gamma}^D}\overline{z} \tag{30}$$

and satisfies

$$\hat{z} \geq \frac{(1-\gamma)\left((1-\delta)^{\alpha-1}(1+\gamma)-1\right)}{\gamma\left(1-\gamma(1-\delta)^{\alpha-1}\right)}\overline{z} > \overline{z} \tag{31}$$

Proof. See Appendix. □

For $\delta > 0$, $k^P(z)$ and $k^D(z)$ differ for two reasons. First, $\omega^D > \omega^P$, which captures the fact that the presence of depreciation creates an incentive to overinvest. Second and as discussed above, $\tilde{\gamma}^D < \tilde{\gamma}^P$, which captures the fact that the firm discounts future payoffs in the event of no trade at a lower rate than the planner and therefore puts too low of a weight on current versus future expected productivity. This leads to overinvestment for low z and underinvestment for high z. If $\gamma(1-\delta)^{\alpha-1} > 1$, the depreciation channel is so strong that the firm always overinvests.[17] Otherwise, the firm overinvests for z below some threshold $\hat{z}$, and underinvests for z above the threshold. Inequality (31) confirms that, as long as $\delta > 0$, this threshold is strictly larger than $\overline{z}$. The presence of depreciation makes it more likely that, in the future, the firm will be a buyer of capital, thus increasing the set of productivity realizations for which the firm overinvests.

To understand how the lower bound in (31) is constructed, notice that by Proposition 6, a sufficient condition for overinvestment is

$$\frac{\gamma^t z + (1-\gamma^t)\overline{z}}{\gamma^{t+1} z + (1-\gamma^{t+1})\overline{z}} < (1-\delta)^{\alpha-1} \;\; \forall t \geq 1. \tag{32}$$

The left-hand side is increasing in z, and increasing in t for any $z > \overline{z}$. This implies that a sufficient condition for overinvestment is $z > z^*$, where z^* solves

$$\frac{\gamma z^* + (1-\gamma)\overline{z}}{\gamma^2 z^* + (1-\gamma^2)\overline{z}} = (1-\delta)^{\alpha-1} \tag{33}$$

Solving for z^* then yields the lower bound in (31). Notice that this lower bound depends on γ and δ but not λ or ϕ. This is useful for two reasons. First, it confirms the intuition of Proposition 6 that the direction of the inefficiency depends on the tradeoff between the

[17] Intuitively, for underinvestment to ever occur, there must be a sufficiently high probability that a high-productivity firm will eventually become low-productivity; i.e. γ must be sufficiently small. The higher δ, the lower γ must be for this to occur.

productivity process and depreciation. Second, it may be easier in practice to obtain reliable estimates for γ and δ than for λ and ϕ. Moreover, (33) shows that the threshold for over-investment depends on the concavity of the production function: the lower α, the stronger the effect of depreciation on the marginal product of capital and therefore the lower the threshold for overinvestment.

We close this section by characterizing the optimal capital choice for extreme values of the productivity persistence parameter, γ. The parameter determines how likely productivity is to change over time, making it a key determinant of the gains from trade. Recall that Corollary 4 in Section 2 established that the degree of inefficiency was non-monotonic in δ (which, in the deterministic model, is the only determinant of the gains from trade). The following result establishes, analogously, that the degree and direction of inefficiency is also non-monotonic in γ.

Corollary 9. *If either $\gamma = 0$ or $\gamma = 1$, then $k^D(z) \geq k^P(z)$, with equality if and only if $\delta = 0$.*

Proof. This is immediate from Proposition 6 and the fact that, both for $\gamma = 0$ and $\gamma = 1$,

$$\frac{\xi_t(z)}{\xi_{t+1}(z)} = 1 \leq \frac{f'\left((1-\delta)^t k\right)}{f'\left((1-\delta)^{t-1} k\right)} \; \forall t, k,$$

with equality if and only if $\delta = 0$. $\square$

When $\gamma = 0$, current productivity is unimportant in determining the optimal capital choice for future periods. When $\gamma = 1$, a firm expects its productivity to remain at its current level forever, and hence does not anticipate selling or buying capital as a result of changing productivity. In either case, depreciation becomes the only motive for trade. In particular, all firms overinvest unless $\delta = 0$, as established in Section 2.[18] Thus, underinvestment can only occur for intermediate values of persistence.

[18] This result holds, in fact, for more general stochastic processes, since $\frac{\xi_t(z)}{\xi_{t+1}(z)} = 1$ whenever productivity shocks are either i.i.d. or permanent.

4 Implications

The dynamic inefficiency of the decentralized equilibrium has important implications for the distribution of capital across firms; the sensitivity of investment to business cycle shocks; and capital taxation.

4.1 Distribution of capital across firms

Propositions 7 and 8 state that in the event of trade, the capital stock of high-productivity firms is inefficiently small and the capital stock of low-productivity firms is inefficiently high. This implies the following result about the distribution of capital across firms.

Corollary 10. *For $\delta = 0$, or $\delta \geq 0$ and $f(k) = k^{\alpha}$ with $\alpha \in (0,1)$, the distribution of capital across firms in the decentralized equilibrium is inefficiently concentrated.*

Proof. The proof follows immediately from $\tilde{\gamma}^D < \tilde{\gamma}^P$ in Propositions 7 and 8. □

Intuitively, if either $\delta = 0$ or the production function is of the power form, optimal k depends on a weighted average of z and $\overline{z}$. The weight on z is smaller in the decentralized equilibrium than in the planner's solution (i.e. $\tilde{\gamma}^D < \tilde{\gamma}^P$) because firms discount future payoffs in the event of no trade at a lower rate than the planner. Hence, the firms put too low of a weight on current versus future expected productivity and the distribution of capital is less disperse than the one under the planner's solution. Similarly, because firms take less extreme trading positions than the planner, it can be shown that trade volume in the decentralized equilibrium is inefficiently small.

These results are consistent with Lagos and Rocheteau (2009) who show that an increase in dealer bargaining power results in lower dispersion of asset holdings and lower trade volume. In contrast, our results show that both dispersion of asset holdings and trade volume are *inefficiently* low relative to the social optimum.

4.2 Sensitivity of investment to business cycle shocks

The above results suggest, more generally, that investment is insufficiently responsive to productivity shocks. Below we confirm this intuition by deriving the elasticity of the optimal capital choice with respect to z and showing that it is lower in the decentralized equilibrium than in the planner's solution. In other words, firms' investment choices underreact to productivity shocks.

Corollary 11.

1. *Suppose $\delta = 0$. Then the elasticity of k^D evaluated at $z = \overline{z}$ is smaller than the elasticity of k^P:*
$$\left.\frac{dk^D}{dz}\frac{z}{k^D}\right|_{z=\overline{z}} < \left.\frac{dk^P}{dz}\frac{z}{k^P}\right|_{z=\overline{z}} \tag{34}$$

2. *Suppose $\delta \geq 0$, and the production function has the form $f(k) = k^{\alpha}$, $\alpha \in (0,1)$. Then the elasticity of k^D is smaller than the elasticity of k^P for any z:*
$$\frac{dk^D}{dz}\frac{z}{k^D} < \frac{dk^P}{dz}\frac{z}{k^P} \;\forall z \tag{35}$$

Proof. See Appendix. □

The intuition for this result is very similar to the one above. One difference is that for general δ, we can unambiguously sign the difference in elasticities only for the case of a power production function because in this case, the elasticities are independent of k.

The result has potentially important implications for business cycle analysis. Suppose that productivity z is an aggregate state instead of an idiosyncratic firm-specific state as assumed above. Then, in response to a positive productivity shock, the increase in capital demand is inefficiently small. In a dynamic stochastic general equilibrium model, this results in a less amplified and less persistent response of aggregate variables than under the planner's solution; i.e. business cycles fluctuations are inefficiently small.

4.3 Capital taxation

Since the decentralized equilibrium is constrained inefficient, a natural question is whether a tax or subsidy can improve welfare. In what follows, we establish that a *regressive* tax on capital restores efficiency. Intuitively, this result obtains for the same reason as the insufficient dispersion of capital across firms and the insufficient sensitivity of investment to business cycle shocks: the firm does not put enough weight on current productivity in its optimal decision. We also show that the proposed tax is generally distinct from an investment tax, or a wealth tax.

The capital tax we propose takes the particular form that it is only imposed in the event of trade; i.e. when a firm matches and increases its capital to k', it is subject to a tax $\tau(k')$. The firm's problem, previously given by (20), is therefore

$$v(k,z) = zf(k) + \lambda \max_{k'} \left(\beta \sum_{z'} \pi(z'|z) v(k',z') - \rho(k',k,z)(k' - (1-\delta)k) - \tau(k') \right) + (1-\lambda)\beta \sum_{z'} \pi(z'|z) v((1-\delta)k, z') \tag{36}$$

It is easy to check that the bargained price of capital, previously given by (21), is now

$$\rho(k,k',z) = 1 + \phi \left[\sum_{z'} \pi(z'|z) \frac{\beta v(k',z') - \beta v((1-\delta)k, z') - \tau(k')}{k' - (1-\delta)k} - 1 \right]. \tag{37}$$

As a result, the firm's value function becomes

$$\begin{aligned} v(k,z) &= zf(k) + \lambda(1-\phi)(1-\delta)k \\ &\quad + \lambda(1-\phi) \max_{k'} \left(\beta \sum_{z'} \pi(z'|z) v(k',z') - k' - \tau(k') \right) \\ &\quad + (1 - \lambda(1-\phi))\beta \sum_{z'} \pi(z'|z) v((1-\delta)k, z') \end{aligned} \tag{38}$$

Comparison of (38) with (22) shows that the firm's problem with capital taxation differs from the problem without taxation only by the cost of capital for the firm-dealer pair, which is now $k' + \tau(k')$ instead of k'. The firm's optimal capital choice is therefore determined by the first-order condition

$$1 + \tau'(k') = \beta \sum_{t=0}^{\infty} \left(\beta(1-\hat{\lambda})(1-\delta) \right)^t \left(\xi_{t+1}(z) f'((1-\delta)^t k') + \hat{\lambda}(1-\delta) \right), \tag{39}$$

with $\hat{\lambda} = \lambda(1-\phi)$ as before.

We are interested in finding a marginal tax rate function $\tau'(k')$ such that, for each z, the

k' solving (39) coincides with $k^P(z)$ which is given by (19). Such a function exists. Since (19) gives a one-to-one relationship between z and $k^P(z)$, the solution to (39) with k' replaced by $k^P(z)$ gives a marginal tax schedule as a function of z; i.e. $\tau' = \tau'(z)$. To express this marginal tax schedule as a function of k' instead, define $z^P(k') = (k^P)^{-1}(k')$ as the inverse of $k^P(z) = k'$. Then, for each k', the required marginal tax schedule is the solution to (39) with z replaced by $z^P(k')$.

Following a similar logic to Propositions 7 and 8, we can explicitly characterize the required marginal tax rate for any production function when $\delta = 0$, and for the power production function when $\delta > 0$. For both cases, we show that the optimal tax is regressive, in other words, $\tau'(k)$ is decreasing in k.

Proposition 12.

1. *Suppose that $\delta = 0$. The marginal tax implementing the efficient capital allocation is given by*

$$\tau'(k') = \frac{1-\beta}{1-\beta(1-\lambda(1-\phi))}\left(\frac{\tilde{\gamma}^D z^P(k') + \left(1-\tilde{\gamma}^D\right)\bar{z}}{\tilde{\gamma}^P z^P(k') + \left(1-\tilde{\gamma}^P\right)\bar{z}} - 1\right), \tag{40}$$

   where $\tilde{\gamma}^P$ and $\tilde{\gamma}^D$ are given by (24) and (25), and $z^P(k')$ is the solution to $k^P(z) = k'$. The tax is regressive: $\tau''(k') < 0$. Furthermore, $\tau'(k') > 0$ for $k'^P(\bar{z})$ and $\tau'(k') < 0$ for $k'^P(\bar{z})$.

2. *Suppose that $\delta \geq 0$ and the production function has the form $f(k) = k^\alpha$. The marginal tax implementing the efficient capital allocation is given by*

$$\tau'(k') = \frac{1-\beta}{1-\beta(1-\lambda(1-\phi))(1-\delta)}\left(\frac{\omega^D\left(\tilde{\gamma}^D z^P(k') + \left(1-\tilde{\gamma}^D\right)\bar{z}\right)}{\omega^P\left(\tilde{\gamma}^P z^P(k') + \left(1-\tilde{\gamma}^P\right)\bar{z}\right)} - 1\right), \tag{41}$$

   where ω^P, $\tilde{\gamma}^P$ and ω^D, $\tilde{\gamma}^D$ are given by (27) and (29), and $z^P(k')$ is the solution to $k^P(z) = k'$. The tax is regressive: $\tau''(k') < 0$. If $\gamma(1-\delta)^{\alpha-1} > 1$, then $\tau'(k') > 0$ for all k'. If $\gamma(1-\delta)^{\alpha-1} \leq 1$, then $\tau'(k') > 0$ for $k'^P(\hat{z})$ and $\tau'(k') < 0$ for $k'^P(\hat{z})$, where $\hat{z}$ is given by (30).

Proof. See Appendix. $\square$

The message of Proposition 12 is two-fold. First, the optimal tax is regressive since firms' capital choices are insufficiently responsive to productivity without the tax. Second, when the depreciation channel always dominates, the tax should always be positive. Otherwise, the tax should be positive at low capital levels and negative (i.e. a capital subsidy) at high capital levels, since low-productivity firms are overinvesting and high-productivity firms are underinvesting.

A natural question to ask is how the proposed capital tax compares to alternative taxes such as an investment tax, or a wealth tax. Consider first a tax on investment $x = k' - (1-\delta)k$. Such a tax cannot restore the planner's allocation because what matters for the inefficiency is not the *additional* capital purchased but the *total* capital that the firm wants to have as an outside option in future bargaining.

Consider next a wealth tax; i.e. a tax on the firm's capital stock independent of whether trade occurs or not. In this case, the first-order condition for the firm's optimal capital choice in the event of trade becomes

$$1 = \beta \sum_{t=0}^{\infty} \left(\beta(1-\hat{\lambda})(1-\delta)\right)^t \left(\xi_{t+1}(z) f'((1-\delta)^t k') - \tau'\left((1-\delta)^t k'\right) + \hat{\lambda}(1-\delta)\right). \quad (42)$$

For $\delta > 0$, wealth depends on whether trade occurs in the future. This makes the problem intractable because it is no longer possible to show that there is a unique $\tau' = \tau'(z)$ that can be mapped into $\tau'(k') = \tau'(z^P(k'))$. Hence, it is not clear whether a unique marginal tax function exists and, if it exists, what its properties are.[19]

For $\delta = 0$, by contrast, wealth is constant and the present value of marginal taxes in the event of no trade can be expressed in closed-form. Hence, the same techniques as above can be applied to establish that a regressive tax establishes efficiency. This is important, because the no-depreciation case encompasses the entire literature on financial asset trade in OTC markets with unrestricted asset holdings, starting with Lagos and Rocheteau (2009). Our results thus imply that a regressive wealth tax restores the efficient allocation in this

[19] For a tax on income, $zf(k)$, the problem becomes intractable for very much the same reason as for the wealth tax when $\delta > 0$. The firm's first-order condition contains an infinite sequence of marginal tax rates, one for each future marginal product of capital in the event of no trade. As a result, it is no longer possible to argue that there is a unique $\tau' = \tau'(z)$ that can be mapped into a $\tau'(k') = \tau'(z^P(k'))$.

environment.

5 Conclusion

This paper studies how bargaining in capital markets with trading frictions distorts market participants' incentives and leads to inefficient equilibrium outcomes. The key result coming out of our analysis is that if firms anticipate buying capital in the future, they have a dynamic incentive to accumulate more than the efficient level of capital today. This increases the outside option in negotiations with future dealers, thereby lowering the price at which firms can purchase capital in the future. Conversely, anticipations of capital sales in the future lead firms to accumulate an inefficiently low level of capital today.

The results highlight the importance of capital depreciation in determining the direction of inefficiency. In the absence of depreciation, high-productivity firms underinvest while low-productivity firms overinvest. The presence of depreciation increases the set of productivity realizations for which firms overinvest, because it raises the chance that any firm will become a buyer of capital. A similar argument can be made for other channels that create an incentive to buy capital in the future, such as trend growth in productivity. This suggests that the dynamic investment distortion is greater in faster-growing economies, an interesting extension that we leave for future research.

Our results imply that the dispersion of capital holdings is too low relative to the social optimum, and that investment is insufficiently responsive to shocks. This result serves as a cautionary note against interpreting high volatility or dispersion of asset holdings in decentralized markets as a symptom of inefficiency: exactly the opposite is true here. Moreover, our analysis provides a novel rationale for capital taxation and, more generally, regulation in decentralized capital markets.

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A Proofs

Proof of Proposition 1. First, observe that if Υ takes the form in (3), then it inherits the following properties of f: it is continuous, twice-differentiable, strictly increasing, and strictly concave. As a result, the optimal k' is indeed given by the first-order condition (5) and, in particular, does not depend on k. Since we have restricted k to lie in $[0, \overline{k}]$, $f\left((1-\delta)^t k\right)$ is bounded for any t, and hence $\lim_{t\to\infty} (\beta(1-\lambda))^t \left[f\left((1-\delta)^t k\right) + \lambda(1-\delta)^{t+1} k\right] = 0$; the sum in (3) is therefore well-defined. Second, the mapping defined in (2) maps functions of the form (3) into other functions of this form, and the set of such functions is closed. Since the mapping is a contraction, the unique fixed point must have the form (3), by Corollary 1 to the Contraction Mapping Theorem in Stokey et al. (1989). $\square$

Proof of Lemma 2. Multiplying (5) by $\frac{1-\beta(1-\lambda)(1-\delta)}{\beta}$ and subtracting $\lambda(1-\delta)$ from each side, we see that k^P is determined by $\Gamma\left(\lambda, k^P\right) = \frac{1}{\beta} - (1-\delta)$, where

$$\Gamma\left(\lambda, k^P\right) \equiv (1-\beta(1-\lambda)(1-\delta)) \left[\sum_{t=0}^{\infty} (\beta(1-\lambda)(1-\delta))^t f'\left((1-\delta)^t k^P\right)\right] \tag{43}$$

By the implicit function theorem, $\partial k^P/\partial\lambda = -\Gamma_\lambda/\Gamma_k$, with

$$\Gamma_k = (1-\beta(1-\lambda)(1-\delta)) \left[\sum_{t=0}^{\infty} \left(\beta(1-\lambda)(1-\delta)^2\right)^t f''\left((1-\delta)^t k^P\right)\right] \tag{44}$$

and

$$\Gamma_\lambda = \sum_{t=1}^{\infty} t(\beta(1-\delta))^t (1-\lambda)^{t-1} \left(f'\left((1-\delta)^{t-1} k^P\right) - f'\left((1-\delta)^t k^P\right)\right). \tag{45}$$

By the strict concavity of f, $f''\left((1-\delta)^t k\right) < 0$ and $f'\left((1-\delta)^{t-1} k\right) < f'\left((1-\delta)^t k\right)$ for any k as long as $\delta \in (0,1)$. Hence, $\Gamma_k < 0$ and $\Gamma_\lambda \leq 0$, which in turn implies $\partial k/\partial\lambda \leq 0$, strict as long as $\delta \in (0,1)$. $\square$

Proof of Corollary 4. For $\delta = 0$, both the planner's solution in (5) and the decentralized solution in (14) reduce to $1 = \beta(f'(k') + 1)$, which is independent of the match probability

λ. For $\delta = 1$, both the planner's solution in (5) and the decentralized solution in (14) reduce to $1 = \beta f'(k')$, which is likewise independent of λ. Thus, in both cases, the equilibrium coincides with the planner's solution, independently of ϕ. $\square$

Proof of Lemma 5. The proof is similar to the proof of Lemma 2. We can write (19) as $\Gamma\left(\lambda, k^P, z\right) = \frac{1}{\beta} - (1-\delta)$, where

$$\Gamma\left(\lambda, k^P, z\right) \equiv \left(1 - \beta\left(1-\lambda\right)\left(1-\delta\right)\right)\left[\sum_{t=0}^{\infty}\left(\beta\left(1-\lambda\right)\left(1-\delta\right)\right)^t \xi_{t+1}\left(z\right) f'\left(\left(1-\delta\right)^t k^P\right)\right] \quad (46)$$

By the implicit function theorem, $\partial k^P / \partial\lambda = -\Gamma_\lambda / \Gamma_k$, with

$$\Gamma_k = \left(1 - \beta\left(1-\lambda\right)\left(1-\delta\right)\right)\left[\sum_{t=0}^{\infty}\left(\beta\left(1-\lambda\right)\left(1-\delta\right)^2\right)^t \xi_{t+1}\left(z\right) f''\left(\left(1-\delta\right)^t k^P\right)\right] \quad (47)$$

and

$$\Gamma_\lambda = \sum_{t=1}^{\infty} t\left(\beta\left(1-\delta\right)\right)^t \left(1-\lambda\right)^{t-1}\left(\xi_t\left(z\right) f'\left(\left(1-\delta\right)^{t-1} k^P\right) - \xi_{t+1}\left(z\right) f'\left(\left(1-\delta\right)^t k^P\right)\right). \quad (48)$$

By the strict concavity of f, $f''\left(\left(1-\delta\right)^t k\right) < 0$, and so $\Gamma_k < 0$. A sufficient condition for $\Gamma_\lambda < 0$ is

$$\frac{\xi_t\left(z\right)}{\xi_{t+1}\left(z\right)} < \frac{f'\left(\left(1-\delta\right)^t k\right)}{f'\left(\left(1-\delta\right)^{t-1} k\right)} \ \forall t, k$$

Similarly, a sufficient condition for $\Gamma_\lambda > 0$ is

$$\frac{\xi_t\left(z\right)}{\xi_{t+1}\left(z\right)} > \frac{f'\left(\left(1-\delta\right)^t k\right)}{f'\left(\left(1-\delta\right)^{t-1} k\right)} \ \forall t, k$$

This proves the result. $\square$

Proof of Proposition 7. Assume $\delta = 0$. The optimal choice of capital, $k^P\left(z\right)$, solves the first-order condition

$$1=\beta\sum_{t=0}^{\infty}\left(\beta\left(1-\lambda\right)\right)^{t}\left[\left(\gamma^{t+1}z+\left(1-\gamma^{t+1}\right)\overline{z}\right)f'\left(k\right)+\lambda\right] \tag{49}$$

Subtracting $\frac{\beta\lambda}{1-\beta(1-\lambda)}$ from both sides of (49), we obtain

$$\frac{1-\beta}{1-\beta\left(1-\lambda\right)}=\beta f'\left(k\right)\left[\frac{\gamma}{1-\gamma\beta\left(1-\lambda\right)}z+\left(\frac{1}{1-\beta\left(1-\lambda\right)}-\frac{\gamma}{1-\gamma\beta\left(1-\lambda\right)}\right)\overline{z}\right]. \tag{50}$$

Multiplying both sides by $\frac{1-\beta(1-\lambda)}{(1-\beta)f'(k)}$ gives (24). Next, note that $k^{D}\left(z\right)$ solves the same equation as (49) but with λ replaced everywhere by $\lambda\left(1-\phi\right)$. Replacing λ with $\lambda\left(1-\phi\right)$ gives (25). □

Proof of Proposition 8. Assume $f\left(k\right)=k^{\alpha}$. The first-order condition characterizing the optimal k is now

$$1=\beta\alpha k^{\alpha-1}\sum_{t=0}^{\infty}\left(\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}\right)^{t}\left(\gamma^{t+1}z+\left(1-\gamma^{t+1}\right)\overline{z}\right)+\frac{\beta\lambda\left(1-\delta\right)}{1-\beta\left(1-\lambda\right)\left(1-\delta\right)} \tag{51}$$

Subtracting $\frac{\beta\lambda(1-\delta)}{1-\beta(1-\lambda)(1-\delta)}$ from both sides, we get

$$\frac{1-\beta\left(1-\delta\right)}{1-\beta\left(1-\lambda\right)\left(1-\delta\right)}=\alpha k^{\alpha-1}\frac{1}{1-\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}}\left[\tilde{\gamma}^{P}z+\left(1-\tilde{\gamma}^{P}\right)\overline{z}\right], \tag{52}$$

where

$$\tilde{\gamma}^{P}=\frac{\gamma-\gamma\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}}{1-\gamma\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}} \tag{53}$$

Multiplying both sides by $\frac{1-\beta(1-\lambda)(1-\delta)}{1-\beta(1-\delta)}k^{1-\alpha}$ gives (26). Similarly, k^{D} is the solution to (26) but with λ replaced by $\lambda\left(1-\phi\right)$; replacing λ by $\lambda\left(1-\phi\right)$ gives (28).

To prove part 2, first notice, from (26) and (28), that $k^{D}\left(z\right)>k^{P}\left(z\right)$ if and only if

$$\omega^{D}\left(\tilde{\gamma}^{D}z+\left(1-\tilde{\gamma}^{D}\right)\overline{z}\right)>\omega^{P}\left(\tilde{\gamma}^{P}z+\left(1-\tilde{\gamma}^{P}\right)\overline{z}\right), \tag{54}$$

which can be rearranged as

$$\left(\omega^D\tilde{\gamma}^D - \omega^P\tilde{\gamma}^P\right) z < \left(\omega^D\left(1-\tilde{\gamma}^D\right) - \omega^P\left(1-\tilde{\gamma}^P\right)\right)\overline{z} \tag{55}$$

From (27) and (29), we have $\omega^D > \omega^P$ and $\tilde{\gamma}^D < \tilde{\gamma}^P$, which implies that the right-hand side of (55) is always positive. If $\gamma(1-\delta)^{\alpha-1} > 1$, straightforward algebra implies that $\omega^D\tilde{\gamma}^D > \omega^P\tilde{\gamma}^P$ for all z; in this case, the left-hand side of (55) is negative for all z, and therefore $k^D(z) > k^P(z)$ for all z. Otherwise, it is clear from (55) that $k^D(z) > k^P(z)$ if and only if $z < \hat{z}$, with $\hat{z}$ defined by (30). Finally, the argument in the text has already established that inequality (31) is a consequence of Proposition 6, but it can also be verified directly (using tedious algebra) that (30) implies (31). □

Proof of Corollary 11. To prove part 1, assume that $\delta = 0$. From Proposition 7, the elasticity of k^P with respect to z is

$$\frac{z}{k^P}\frac{dk^P}{dz} = -\frac{f'\left(k^P\right)}{k^P f''\left(k^P\right)}\left(\frac{\tilde{\gamma}^P z}{\tilde{\gamma}^P z + \left(1-\tilde{\gamma}^P\right)\overline{z}}\right), \tag{56}$$

where

$$\tilde{\gamma}^P = \frac{\gamma - \gamma\beta(1-\lambda)}{1-\gamma\beta(1-\lambda)} \tag{57}$$

Similarly, k^D is the solution to (24) but with λ replaced by $\lambda(1-\phi)$; therefore, the elasticity of k^D with respect to z is

$$\frac{z}{k^D}\frac{dk^D}{dz} = -\frac{f'\left(k^D\right)}{k^D f''\left(k^D\right)}\left(\frac{\tilde{\gamma}^D z}{\tilde{\gamma}^D z + \left(1-\tilde{\gamma}^D\right)\overline{z}}\right), \tag{58}$$

where

$$\tilde{\gamma}^D = \frac{\gamma - \gamma\beta(1-\lambda(1-\phi))}{1-\gamma\beta(1-\lambda(1-\phi))} < \frac{\gamma - \gamma\beta(1-\lambda)}{1-\gamma\beta(1-\lambda)} = \tilde{\gamma}^P \tag{59}$$

Finally, from Proposition 7, $k^D = k^P$ at $z = \overline{z}$. This proves the result.

To prove part 2, assume $f(k) = k^\alpha$. From Proposition 8, the elasticity of k^P with respect to z is

$$\frac{z}{k^P}\frac{dk^P}{dz} = \frac{1}{1-\alpha}\left(\frac{\tilde{\gamma}^P z}{\tilde{\gamma}^P z + \left(1-\tilde{\gamma}^P\right)\overline{z}}\right), \tag{60}$$

Similarly, k^D is the solution to (26) but with λ replaced by $\lambda(1-\phi)$; therefore, the elasticity of k^D with respect to z is

$$\frac{z}{k^D}\frac{dk^D}{dz} = \frac{1}{1-\alpha}\left(\frac{\tilde{\gamma}^D z}{\tilde{\gamma}^D z + \left(1-\tilde{\gamma}^D\right)\overline{z}}\right), \tag{61}$$

where

$$\tilde{\gamma}^D = \frac{\gamma - \gamma\beta\left(1-\lambda\left(1-\phi\right)\right)\left(1-\delta\right)^{\alpha}}{1-\gamma\beta\left(1-\lambda\left(1-\phi\right)\right)\left(1-\delta\right)^{\alpha}} < \frac{\gamma-\gamma\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}}{1-\gamma\beta\left(1-\lambda\right)\left(1-\delta\right)^{\alpha}} = \tilde{\gamma}^P \tag{62}$$

This proves the result. Note that, for a power production function, this holds for any z, because $-\frac{f'(k)}{kf''(k)} = \frac{1}{1-\alpha}$, independently of k. $\square$

Proof of Proposition 12. The expressions in (40) and (41) are obtained by applying the first-order condition (39) and using the same simplification as in the proofs of Propositions 7 and 8. Next, when $\delta = 0$, the marginal tax is positive if and only if $\tilde{\gamma}^D z^P(k') + \left(1-\tilde{\gamma}^D\right)\overline{z} > \tilde{\gamma}^P z^P(k') + \left(1-\tilde{\gamma}^P\right)\overline{z}$, which, as shown in Proposition 7, is true if and only if $z^P(k') < \overline{z}$. Similarly, when $f(k) = k^{\alpha}$, the marginal tax is positive if and only if $\omega^D\left(\tilde{\gamma}^D z^P(k') + \left(1-\tilde{\gamma}^D\right)\overline{z}\right) > \omega^P\left(\tilde{\gamma}^P z^P(k') + \left(1-\tilde{\gamma}^P\right)\overline{z}\right)$. As shown in Proposition 8, if $\gamma(1-\delta)^{\alpha-1}$, this is true for all $z^P(k')$; otherwise, this is true if and only if $z^P(k') < \hat{z}$. It remains to verify that the tax is regressive. Since $\tilde{\gamma}^D < \tilde{\gamma}^P$, the fraction

$$\frac{\tilde{\gamma}^D z + \left(1-\tilde{\gamma}^D\right)\overline{z}}{\tilde{\gamma}^P z + \left(1-\tilde{\gamma}^P\right)\overline{z}} \tag{63}$$

is decreasing in z. Since $z^P(k')$ is increasing in k', this implies that $\tau'(k')$ is decreasing in k'. $\square$