Graph complexes and deformation quantization of Lie bialgebroids

2023 Gone Fishing

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Talk based on K. M. SIGMA 18 (2022) 020

Via Zoom, 2023/03/19

Context in a nutshell

- ## Deformation quantization

  A research program aiming at providing a rigorous framework for quantization. The main idea lies in the introduction of a formal deformation parameter $\hbar$ allowing to distinguish between three layers of structures:

	Classical $\hbar^0$	**Semi-classical $\hbar^1$**	**Quantum $\hbar^{n>1}$**
Poisson manifolds	Algebra of functions $(\mathscr{C}^\infty(\mathscr{M}), \cdot)$	Poisson structure π	Star product $(\mathscr{C}^\infty(\mathscr{M}), *_\hbar)$
Lie bialgebras	Vector space $\mathfrak{g}$	Lie bialgebra $([\cdot,\cdot]_\mathfrak{g}, \delta_\mathfrak{g})$	Quantum group $(S(\mathfrak{g}), *_\hbar, \Delta_\hbar)$

- ## Formality theorem

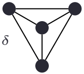
  The most important result in deformation quantization. Due to Kontsevich 97', it provides a Lie_∞ quasi-isomorphism:

  $$\mathcal{U} : \mathcal{T}_{\text{poly}} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}$$

  between

  - $\mathcal{T}_{\text{poly}}$ the Schouten graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^m$ (Poisson structures)
  - $\mathcal{D}_{\text{poly}}$ the Hochschild differential graded Lie algebra of multidifferential operators on $\mathbb{R}^m$ (Star products)

- ## Graph complexes GC

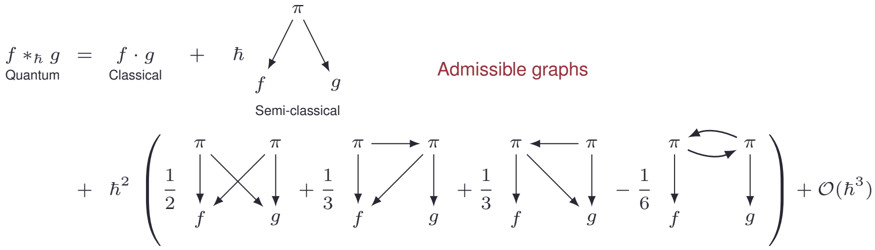
  A (family of) of cochain complexes made of graphs in which the differential acts by blowing up edges, *e.g.* δ  $= 0$

- ## Grothendieck–Teichmüller group GRT_1

  An important (and still vastly mysterious) infinite-dimensional group — introduced by V. Drinfel'd (in its pro-unipotent version) based on ideas of A. Grothendieck — acting on a wide variety of objects in many different mathematical contexts (*e.g.* the Kashiwara–Vergne conjecture, multiple zeta values, rational homotopy of the $\mathbf{E}_2$-operad, *etc.*).

Deformation quantization and formality

- Let $(\mathbb{R}^m, \pi)$ be a Poisson manifold. The Kontsevich's star product formula reads at first orders:



Admissible graphs

Higher orders

- The resulting quantization formula is:
  - Associative *i.e.* $(f \ast_\hbar g) \ast_\hbar h = f \ast_\hbar (g \ast_\hbar h)$
  - Universal *i.e.* is valid for any Poisson bivector π, in any (finite) dimension $m < \infty$
  - Transcendental *i.e.* involves integrals over configuration spaces of points

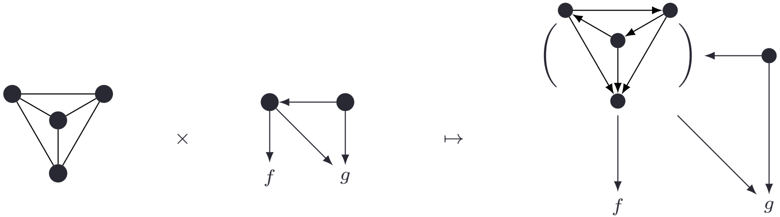
(or more generally, Drinfel'd associators)

- Kontsevich's quantization formula can be interpreted as the Feynman diagram expansion associated with the BV quantization of the Poisson σ-model (with source of $d = 2$). Cattaneo, Felder 99'

Formality and graph complexes

- Kontsevich graphs (from GC) act on admissible graphs (from the quantization formula):

$$\mathsf{GC} \times \{\text{Admissible graphs}\} \to \{\text{Admissible graphs}\}$$



- Whenever the graph belongs to $H^0(\mathsf{GC}_2)$, the associativity of the star-product is preserved.
- More generally, the exponential group $\exp\left(H^0(\mathsf{GC}_2)\right)$ acts regularly on the space of universal formality morphisms Dolgushev 11'.

Graph complexes and GRT_1

- The group GRT_1 satisfies $\mathsf{GRT}_1 = \exp(\mathfrak{grt}_1)$ with $\mathfrak{grt}_1$ the Grothendieck–Teichmüller algebra.
- As shown in Willwacher 10':

$$H^0(\mathsf{GC}_2) \simeq \mathfrak{grt}_1 \ .$$

- Overall, we conclude that GRT_1 acts regularly on the space of universal formality morphisms *i.e.* formality maps form a GRT_1-torsor.

$$\mathcal{U} : \mathcal{T}_{\text{poly}} \xrightarrow[\circlearrowleft \mathsf{GRT}_1]{\sim} \mathcal{D}_{\text{poly}}$$

- This action of GRT_1 can be traced back to an action of the graph complex GC_2 on $\mathcal{T}_{\text{poly}}$:

$$\mathsf{GC}_2 \to \mathsf{CE}(\mathcal{T}_{\text{poly}}).$$

- The cohomology of the graph complex provides information on aspects of the deformation quantization problem:
  - **Existence**: Obstructions to the existence of universal formality maps live in $H^1(\mathsf{GC}_2) \overset{?}{\simeq} \mathbf{0}$.
  - **Classification**: The space of universal formality maps is classified by $H^0(\mathsf{GC}_2) \simeq \mathfrak{grt}_1$.

Summary of this talk

- Universal solutions to deformation quantization problems are characterised by the cohomology of suitable graph complexes (denoted collectively GC):
    - **Existence**: Obstructions to the existence of universal solutions live in $H^1(\mathrm{GC})$.
    - **Classification**: The space of universal solutions is classified by $H^0(\mathrm{GC})$.

- This approach has been successfully applied to the following deformation quantization problems:
    - Poisson manifolds $(\dim < \infty)$ *cf.* Kontsevich 93', Willwacher 10'
    - Poisson manifolds $(\dim = \infty)$ *cf.* Penkava–Vanhaecke 98', Shoikhet 08', Willwacher 13'
    - Lie bialgebras *cf.* Merkulov–Willwacher 15', 16'

- The aim of this talk is to add two threads to this on-going story:
    - Lie bialgebroids
    - Quasi-Lie bialgebroids

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  - **Existence**: Obstructions to the existence of universal solutions live in $H^1(\mathsf{GC})$.
  - **Classification**: The space of universal solutions is classified by $H^0(\mathsf{GC})$.
- Deformation quantization problems can be partitioned into different categories according to the cohomology of the graph complex acting on them:

	$H^1(\mathsf{GC}) \overset{?}{\simeq} \mathbf{0}$ (Yes-go) $H^0(\mathsf{GC}) \simeq \mathfrak{grt}_1$	$H^1(\mathsf{GC}) \simeq \mathbb{K}$ (No-go) $H^0(\mathsf{GC}) \simeq \mathbf{0}$
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	Quasi-Lie bialgebroids	Lie bialgebroids

Lie bialgebroids Mackenzie, Xu 94'

- A natural candidate to unify Poisson manifolds and Lie bialgebras is given by the notion of Lie bialgebroid, *i.e.* a vector bundle $E \overset{\pi}{\to} \mathscr{M}$ endowed with two Lie algebroid structures:
  - $(\rho, [\cdot,\cdot]_E)$ on E
  - $(R, [\cdot,\cdot]_{E^*})$ on E^*

  satisfying a (quadratic) compatibility condition.

  Examples: Lie bialgebras, tangent bundle $T\mathscr{M}$ and cotangent bundle $T^*\mathscr{M}$ of a Poisson manifold

- Lie bialgebroids can be naturally recast within graded geometry as Hamiltonian functions on the graded symplectic manifold $T^*[2]E[1]$ with symplectic 2-form $\omega = d\underset{0}{x^\mu} \wedge d\underset{2}{p_\mu} + d\underset{1}{\xi^a} \wedge d\underset{1}{\zeta_a}$.
- The corresponding graded Poisson bracket is called the "big-bracket" and denoted $\{\cdot,\cdot\}_\omega$.
- The Hamiltonian function reads D. Roytenberg 02, Y. Kosmann–Schwarzbach 05':

$$\mathscr{H} = \rho_a{}^\mu(x)\,\xi^a p_\mu - \frac{1}{2} f_{[ab]}{}^c(x)\,\xi^a\xi^b\zeta_c + R^{a|\mu}(x)\,\zeta_a p_\mu - \frac{1}{2} C_c{}^{[ab]}(x)\,\zeta_a\zeta_b\xi^c$$

  where
  - (ρ, f) are the structure constants of the Lie algebroid structure on E.
  - (R, C) " " " E^*.

- Imposing $\left\{\mathscr{H}, \mathscr{H}\right\}_\omega = 0$, one recovers the compatibility conditions between the two structures.

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Examples

- Lie bialgebras $\quad \mathscr{H} = -\frac{1}{2} f_{[ab]}{}^c\,\xi^a\xi^b\zeta_c - \frac{1}{2} C_c{}^{[ab]}\,\zeta_a\zeta_b\xi^c$
- Poisson manifolds $\quad \mathscr{H} = \xi^\mu p_\mu + \pi^{\mu\nu}(x)\,\zeta_\mu p_\nu - \frac{1}{2}\partial_\lambda \pi^{\mu\nu}(x)\,\zeta_\mu\zeta_\nu\xi^\lambda$

Lie bialgebroids Mackenzie, Xu 94'

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- Adding terms to the Hamiltonian function allows to deform the notion of Lie bialgebroid:
  - quasi-Lie bialgebroid: $\frac{1}{6}\psi_{[abc]}(x)\,\xi^a\xi^b\xi^c$
  - Lie-quasi bialgebroid: $\frac{1}{6}\varphi^{[abc]}(x)\,\zeta_a\zeta_b\zeta_c$

  Examples: quasi-Lie bialgebras, twisted Poisson manifolds

Deformation quantization of Lie bialgebroids

- The deformation quantization problem for Lie bialgebroids is due to Xu 97'. The associated quantum object is an (associative) bialgebroid unifying the notions of star product and bialgebra.
- The quantization problem for Lie bialgebroids reads

	Classical $\hbar^0$	**Semi-classical** $\hbar^1$	**Quantum** $\hbar^{n>1}$
Lie bialgebroids	Vector bundle $E \xrightarrow{\pi} \mathscr{M}$	Lie bialgebroid $(\rho, [\cdot,\cdot]_E, R, [\cdot,\cdot]_{E^*})$	Quantum groupoid $(S(E), *_\hbar, \Delta_\hbar, \alpha_\hbar, \beta_\hbar)$

  and is still open, in full generality, although some particular cases are known to be quantizable

  (Lie bialgebras, Lie bialgebroids associated to Poisson manifolds, triangular Lie bialgebroids, see Xu 97', Calaque 04').

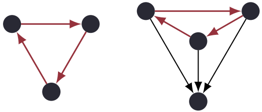
- By analogy with the Poisson and Lie bialgebra cases, one can formulate the two natural conjectures:
  - **Existence**: Every Lie bialgebroid is quantizable as a quantum groupoid. Xu 97'
  - **Classification**: The space of quantizations is a GRT_1-torsor.

- These are hard conjectures and arguably not much progress has been made since their formulation.

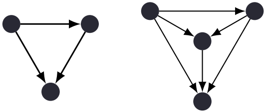
Can graph complexes provide some insights?

Climbing the dimension ladder

- Graphs come in various flavors, depending on the sort of geometric structures they act on:
  - Poisson manifolds ($\dim < \infty$): Directed graphs ($d = 2$)
  - Lie bialgebras: Oriented graphs ($d = 3$) *i.e.* graphs without cycles



Examples of cycle graphs



Examples of oriented graphs

- The cohomology of the graph complexes depend on both d and the number c of oriented colors:

$$H^\bullet(\mathsf{GC}_d^c) \simeq H^\bullet(\mathsf{GC}_{d+1}^{c+1})$$

Willwacher 13', Živković 17'

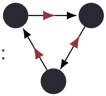
- This allows to yield novel incarnations of familiar structures in higher dimension d:

Examples

$\mathfrak{grt}_1$	$\simeq$	$\times$	$\simeq$	$H^0(\mathsf{GC}_2)$ Poisson manifolds $\dim<\infty$	$\simeq$	$H^0(\mathsf{GC}_3^1)$ Lie bialgebras	$\simeq$	$\dots$
$\mathbb{K}$	$\simeq$	$H^1(\mathsf{GC}_1)$ Moyal star commutator	$\simeq$	$H^1(\mathsf{GC}_2^1)$ Poisson manifolds $\dim=\infty$	$\simeq$	$H^1(\mathsf{GC}_3^2)$?	$\simeq$	$\dots$

Graph actions on Lie bialgebroids

K.M. 22'



- In order to define a graph action on Lie bialgebroids, we need to resort to two-colored graphs:
  - Lie bialgebroids : Two-colored graphs with two oriented directions ($d = 3$, $c = 2$)
  - Quasi-Lie bialgebroids : Two-colored graphs with one oriented direction ($d = 3$, $c = 1$)
- The quantization problem for Lie bialgebroids is akin to the one for Poisson manifolds in $\dim = \infty$.

Theorem

The deformation complex of Lie bialgebroids is endowed with an exotic Lie_∞*-structure deforming non-trivially the so-called "big bracket".*

No-go

- The quantization problem for quasi-Lie bialgebroids is akin to the one for Poisson manifolds in $\dim < \infty$.

Theorem

The Grothendieck–Teichmüller group acts via Lie_∞*-automorphisms on the deformation complex of quasi-Lie bialgebroids.*

Yes-go

Summary and outlook

- Deformation quantization problems can be partitioned into different categories according to the cohomology of the graph complexes acting on them:

	$H^1(\mathsf{GC}) \overset{?}{\simeq} \mathbf{0}$ (Yes-go)	$H^1(\mathsf{GC}) \simeq \mathbb{K}$ (No-go)
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	Quasi-Lie bialgebras	
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- The quantization problem for Lie bialgebroids differs essentially from the Lie bialgebra case:
  1. There is a potential obstruction to the existence of universal quantizations of Lie bialgebroids.
  2. The Grothendieck–Teichmüller group plays no classifying rôle.

- This result allows us to formulate the following conjecture (No-go):

  There are no universal quantizations of Lie bialgebroids as quantum groupoids.

  Settling this question requires a better understanding of the deformation theory of bialgebroids.

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- The quantization problem for quasi-Lie bialgebroids is similar to the the (quasi)-Lie bialgebra case:
  1. There is (conjecturally) no generic obstruction to the existence of universal quantizations.
  2. The Grothendieck–Teichmüller group plays a classifying rôle.
- This result allows us to formulate the following conjecture (Yes-go):

  Given a Drinfel'd associator, one can define a universal quantization of (quasi-)Lie bialgebroids as quasi-quantum groupoids.

Summary and outlook

Thank you!

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Appendices

Graph actions on geometric structures

Graded geometric formulation of Lie bialgebroids and variations

See *e.g.* D. Roytenberg 02, Y. Kosmann–Schwarzbach 05'

- Let $E \overset{\pi}{\to} \mathscr{M}$ be a vector bundle and consider the graded symplectic manifold $T^*[2]E[1]$ with symplectic 2-form $\omega = d\underset{0}{x^\mu} \wedge d\underset{2}{p_\mu} + d\underset{1}{\xi^a} \wedge d\underset{1}{\zeta_a}$.
- The corresponding graded Poisson bracket is called the "big-bracket" and denoted $\{\cdot,\cdot\}_\omega$.
- The most general function of degree 3 on $T^*[2]E[1]$ reads:

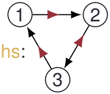
$$\mathscr{H} = \rho_a{}^\mu \xi^a p_\mu - \frac{1}{2} f_{[ab]}{}^c \xi^a \xi^b \zeta_c + R^{a|\mu} \zeta_a p_\mu - \frac{1}{2} C_c{}^{[ab]} \zeta_a \zeta_b \xi^c + \frac{1}{6} \varphi^{[abc]} \zeta_a \zeta_b \zeta_c + \frac{1}{6} \psi_{[abc]} \xi^a \xi^b \xi^c$$

  where $\{\rho, f, R, C, \varphi, \psi\}$ are functions on the base space $\mathscr{M}$.
- Imposing $\left\{\mathscr{H}, \mathscr{H}\right\}_\omega = 0$, the functions $\{\rho, f, R, C, \varphi, \psi\}$ define the following structures:
  - In the most generic case, the structure defined is a proto-Lie bialgebroid.
  - If $\psi_{abc} \equiv 0$, " Lie-quasi bialgebroid.
  - If $\varphi^{abc} \equiv 0$, " quasi-Lie bialgebroid.
  - If $\psi_{abc} \equiv 0$, $\varphi^{abc} \equiv 0$, " Lie bialgebroid.
- To each of these three sub-cases corresponds a graded Poisson subalgebra of $\mathscr{C}^\infty\left(T^*[2]E[1]\right)$, denoted $\mathcal{A}^E_{\text{Lie-quasi}}$, $\mathcal{A}^E_{\text{quasi-Lie}}$ and $\mathcal{A}^E_{\text{Lie}}$, respectively.

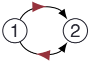
Graph actions on geometric structures

Lie bialgebroids and variations $(d = 3)$

- Let $E \xrightarrow{\pi} \mathscr{M}$ be a vector bundle and consider the graded symplectic manifold $T^*[2]E[1]$ with symplectic 2-form $\omega = \underset{0}{dx^\mu} \wedge \underset{2}{dp_\mu} + \underset{1}{d\xi^a} \wedge \underset{1}{d\zeta_a}$.
- The graded manifold contains two sets of dual coordinates $\{x^\mu, p_\mu\}$ and $\{\xi^a, \zeta_a\}$.

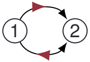


- In order to define a graph action on $\mathscr{C}^\infty(T^*[2]E[1])$, we need to resort to two-colored graphs:
- We define a representation of the 2-colored operad Gra_3^2 on the space of graded functions on $T^*[2]E[1]$ as follows:
- Explicitly, (i) → (j) is mapped to $\dfrac{\partial(i)}{\partial x^\mu}\dfrac{\partial(j)}{\partial p_\mu}$ while (i) ←→ (j) gets mapped to $\dfrac{\partial(i)}{\partial \xi^a}\dfrac{\partial(j)}{\partial \zeta_a}$.

Example 

$(f_1 \otimes f_2) = \dfrac{\partial(1)}{\partial x^\mu}\dfrac{\partial(2)}{\partial p_\mu}\dfrac{\partial(1)}{\partial \xi^a}\dfrac{\partial(2)}{\partial \zeta_a}(f_1 \otimes f_2) = (-1)^{|f_1|}\dfrac{\partial^2 f_1}{\partial x^\mu \partial \xi^a}\dfrac{\partial^2 f_2}{\partial p_\mu \partial \zeta_a}$

Crucial observation: The presence of a red cycle prevents this graph to preserve the deformation complex of quasi-Lie bialgebroids (hence of Lie bialgebroids).

Example Acting on $f_1 \sim \xi\zeta\zeta$ and $f_2 = p\zeta\zeta \in \mathcal{A}^E_{\text{quasi-Lie}}$, we get 

$(f_1 \otimes f_2) = \zeta\zeta\zeta \notin \mathcal{A}^E_{\text{quasi-Lie}}$

Preserving the graded Poisson subalgebras $\mathcal{A}^E_{\text{Lie-quasi}}$, $\mathcal{A}^E_{\text{quasi-Lie}}$ and $\mathcal{A}^E_{\text{Lie}}$ requires orienting colors.

Graph actions on geometric structures

Lie bialgebroids and variations ($d = 3$)

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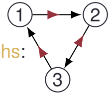
- The graded manifold contains two sets of dual coordinates $\{x^\mu, p_\mu\}$ and $\{\xi^a, \zeta_a\}$.
- In order to define a graph action on $\mathscr{C}^\infty(T^*[2]E[1])$, we need to resort to two-colored graphs:



- We define a representation of the 2-colored operad Gra_3^2 on the space of graded functions $\mathscr{C}^\infty(T^*[2]E[1])$ as follows:
- Explicitly, $(i) \to (j)$ is mapped to $\dfrac{\partial(i)}{\partial x^\mu}\dfrac{\partial(j)}{\partial p_\mu}$ while $(i) \leftrightarrow (j)$ gets mapped to $\dfrac{\partial(i)}{\partial \xi^a}\dfrac{\partial(j)}{\partial \zeta_a}$.

Theorem K.M. 22'

- $\mathscr{C}^\infty(T^*[2]E[1])$ *is endowed with an action of the plain operad* $\mathsf{Gra}_3^{2|0}$.
- $\mathcal{A}^E_{\text{Lie-quasi}}$ “ *black-oriented operad* $\mathsf{Gra}_3^{2|\text{black}}$.
- $\mathcal{A}^E_{\text{quasi-Lie}}$ “ *red-oriented operad* $\mathsf{Gra}_3^{2|\text{red}}$.
- $\mathcal{A}^E_{\text{Lie}}$ “ *bi-oriented operad* $\mathsf{Gra}_3^{2|2}$.

Classification of quantization problems

	$H^1(\mathsf{GC}) \overset{?}{\simeq} \mathbf{0}$ $H^0(\mathsf{GC}) \simeq \mathfrak{grt}_1$	$H^1(\mathsf{GC}) \simeq \mathbb{K}$ $H^0(\mathsf{GC}) \simeq \mathbf{0}$	$H^1(\mathsf{GC}) \simeq \mathbf{0}$ $H^0(\mathsf{GC}) \simeq \mathbb{K}$
$d = 2$	Poisson $(\dim < \infty)$	Poisson $(\dim = \infty)$	
$d = 3$	Lie bialgebras Lie-quasi bialgebras Quasi-Lie bialgebras		Proto-Lie bialgebras
	Lie-quasi bialgebroids Quasi-Lie bialgebroids	Lie bialgebroids	Proto-Lie bialgebroids
			Courant algebroids

Conjectures

$$\left(\mathcal{A}^E_{\mathrm{Lie}}, \{\cdot,\cdot\}^E_\omega\right) \overset{\times}{\dashrightarrow} \left(\mathcal{A}^E_{\mathrm{Lie}}, \theta\right) \xrightarrow{\quad\sim\quad} \left(C^\bullet_{\mathsf{GS}}(\mathcal{O}_E, \mathcal{O}_E), \mu\right)$$

$$\text{Lie bialgebroids} \overset{\times}{\dashrightarrow} \begin{matrix}\text{“Quantizable}\\ \text{Lie bialgebroids”}\end{matrix} \xrightarrow{\quad\sim\quad} \begin{matrix}\text{Quantum}\\ \text{groupoids}\end{matrix}$$

$$\left(\mathcal{A}^E_{\text{Lie-quasi}}, \{\cdot,\cdot\}^E_\omega\right) \xrightarrow[\circlearrowleft \mathrm{GRT}_1]{\sim} \left(\mathcal{A}^E_{\text{Lie-quasi}}, \theta\right) \xrightarrow{\quad\sim\quad} \left(C^\bullet_{\text{quasi}-\mathsf{GS}}(\mathcal{O}_E, \mathcal{O}_E), \mu\right)$$

$$\begin{matrix}\text{Lie-quasi}\\ \text{bialgebroids}\end{matrix} \xrightarrow[\circlearrowleft \mathrm{GRT}_1]{\sim} \begin{matrix}\text{“Quantizable}\\ \text{Lie-quasi bialgebroids”}\end{matrix} \xrightarrow{\quad\sim\quad} \begin{matrix}\text{Quasi-quantum}\\ \text{groupoids}\end{matrix}$$