# Line Integral and Area Without Green

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## Abstract

Certain line integrals of vector fields calculate areas on the plane. This is typically proved using Green's Theorem. In this article, we discuss the geometric meanings of five such line integrals without using Green's Theorem. The first three are from calculus classes. The last two model wheel measures of Amsler's linear and polar planimeters.

## 1 Introduction

Line Integrals of appropriately chosen vector fields can be used to calculate areas of regions on the plane. To be precise, let (1) $C$ be a simple closed, continuously differentiable, and positively-oriented curve on the $xy$-plane, (2) $D$ the open region bounded by $C$, and (3) $\boldsymbol{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ a vector field continuous on $D \cup C$ and continuously differentiable in $D$. The following corollary to Green's Theorem appears in many calculus textbooks around the world. It would be fun to go to your local library to check it out. Let's go!

**Theorem 1.** *If $Q_x - P_y = 1$, then the line integral $\int_C P dx + Q dy$ equals* $\text{Area}(D)$.

*Proof.* $\int_C P dx + Q dy \overset{\text{Green's Theorem}}{=\!=\!=\!=} \iint_D Q_x - P_y dA = \iint_D 1 dA = \text{Area}(D).$ ◻

In this article, we will review five examples of $\int_C P dx + Q dy = \text{Area}(D)$ without taking a detour to Green's Theorem. Instead, we go directly from line integral to area by examining the geometric meaning of $P\Delta x + Q\Delta y$. After discussing three warm-up examples found in many calculus textbooks, we proceed to Amsler's linear and polar planimeters. In contrast with verifying $Q_x - P_y = 1$, which is detachedly symbolic and at times laborious, especially in the case of Amsler's polar planimeter, all geometries discussed can be directly visualized. The method of proof used in the last section may also serve as a preview of some techniques which could be used in the proof of hard versions of Green's and Cauchy's Theorems.

**Remark.** Continuous differentiability is a natural condition to impose on $C$ if it is drawn by hand or a mechanical device. To see it, let $C$ be the trajectory of the tip of a chalk or a point on any other device modeled by its position vector function $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$. Then $\boldsymbol{r}(t)$ satisfies Newton's Second Law of Motion, where $\boldsymbol{r}''(t)$ makes sense. Therefore, $\boldsymbol{r}'(t)$ is continuous. Such $C$ includes piecewise smooth curves with sharp corners where $\boldsymbol{r}'(t) = 0$. In general, Green's Theorem also holds when $C$ is no longer continuously differentiable [12]. Condition (3) can also be weakened [5] beyond Riemann integrability considered in this article.

## 2 Curves drawn by hand in class

**Example 1.** $\int_C x dy$

Geometrically, $x$ is the signed distance from $(x, y)$ to the $y$-axis and $\Delta y$ is a small vertical displacement as $(x, y)$ traces $C$. Thus, $x\Delta y$ is the signed area of a horizontal strip with length $x$ and width $\Delta y$. (See Figure 154 of [2].) If $D$ is a rectangle with sides parallel to the coordinate axes, the top and bottom edges where $\Delta y = 0$ do not contribute to the integral. When $\Delta y$ are taken from the left (Figure 1e) and right (Figure 1a)

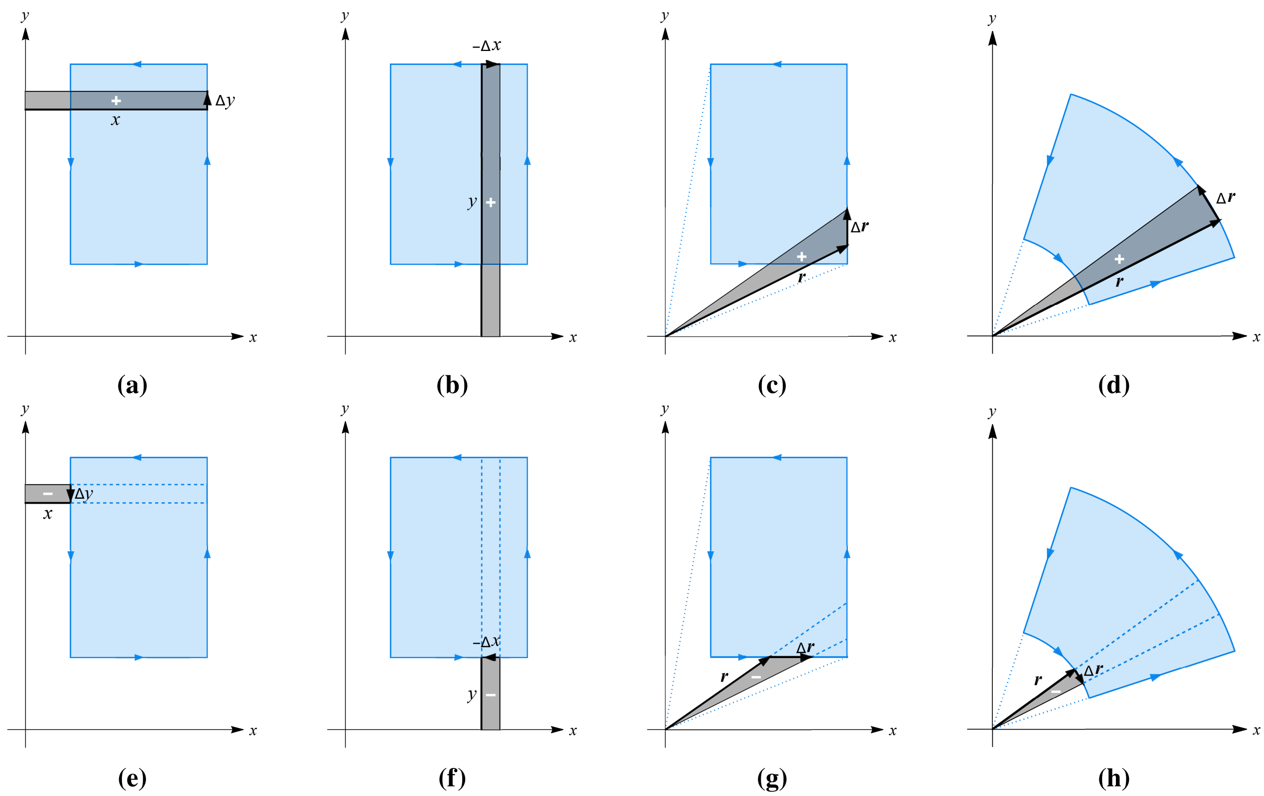



**Figure 1:** *Elementary domains for Examples 1, 2 and 3. The "+/−" signs indicate the signs of the areas.*

edges on the same height, the two $x\Delta y$ add up to the area of the horizontal strip inside the rectangle bounded by the dashed line. Figure 1(a,e) shows the case when both vertical edges of the rectangle are to the right of the $y$-axis. Same result holds when both are to the left of the $y$-axis, and when they are on opposite sides of the $y$-axis. Thus, $\int_C xdy = \text{Area}(D)$ after we take the limit of the Riemann sum.

**Example 2.** $\displaystyle\int_C -ydx$

This line integral is the symmetric image of that in Example 1 under the $x|y$ mirror reflection, but we need to add a negative sign to flip the direction of $C$ back to positive-orientation. Therefore, $\int_C -ydx = \text{Area}(D)$, which can also be verified by Figure 1(b,f) as in Example 1. Readers who have learned single-variable but not multivariable calculus would be at a better position to immediately recognize the geometric meaning of this integral.

**Example 3.** $\displaystyle\frac{1}{2}\int_C -ydx + xdy$

This line integral is the arithmetic mean of those in Examples 1 and 2, but it has a direct geometric meaning as well. The determinant $\begin{vmatrix} \boldsymbol{r} \\ \Delta\boldsymbol{r} \end{vmatrix}$ is the signed area of the parallelogram spanned by the vectors $\boldsymbol{r}$ and $\Delta\boldsymbol{r}$, and thus, $\frac{-y}{2}\Delta x + \frac{x}{2}\Delta y$, which is half of this determinant, is the signed area of a trianglular strip with one vertex at the origin. If $D$ is an annular sector which is centered around the origin, the two straight edges converging to the origin where $\Delta\boldsymbol{r} = 0$ do not contribute to the integral, but $\frac{-y}{2}\Delta x + \frac{x}{2}\Delta y$ for the two $\Delta\boldsymbol{r}$ bounded between the same two radial lines (Figure 1(d,h)) add up to the area of the quadrilateral bounded between the dashed lines inside the region $D$. Thus, $\frac{1}{2}\int_C -ydx + xdy = \text{Area}(D)$ after we take the limit of the Riemann sum. The same holds if $D$ is a rectangle as illustrated in Figure 1(c,g) where $D$ is divided by radial lines.

**Remark.**

1. This line integral is also the imaginary part of $\frac{1}{2}\int_C \bar{z}dz$ (See Section 8.5.2 of of [11]), whose real part $\frac{1}{2}\int_C xdx + ydy$ vanishes as it is of the conservative vector field $\nabla(x^2/4 + y^2/4)$.
2. Generalizing this method to 3D, it can also be shown without using Gauss's Theorem that the surface integral and its discrete formula of the 2-form $\frac{1}{3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$ calculates volumes [3] by interpreting the term in the Riemann sum as the signed volume of the Egyptian pyramid with a parallelogram base and the apex at the origin, which is $\frac{1}{3}$ of the signed volume of the associated parallelepiped as a scalar triple product.
3. That the surface integrals of the simpler forms $xdy \wedge dz$, $ydz \wedge dx$, and $zdx \wedge dy$ also calculate volumes can be similarly visualized as in Examples 1 and 2. Readers who have learned double integral but not vector calculus yet would be at a better position to immediately recognize the geometric meaning of these three integrals.

We call a rectangle with sides parallel to the coordinate axes an elementary region of Type I (Examples 1, 2 and 3), and an annular sector centered around the origin an elementary region of Type II (Example 3). We have shown that each line integral in Examples 1, 2 and 3 equals area for their elementary regions, but it surely also holds for general regions:

**Theorem 2.** *The line integrals in Examples 1, 2 and 3 all calculate areas of general regions $D$ described in (1) and (2).*

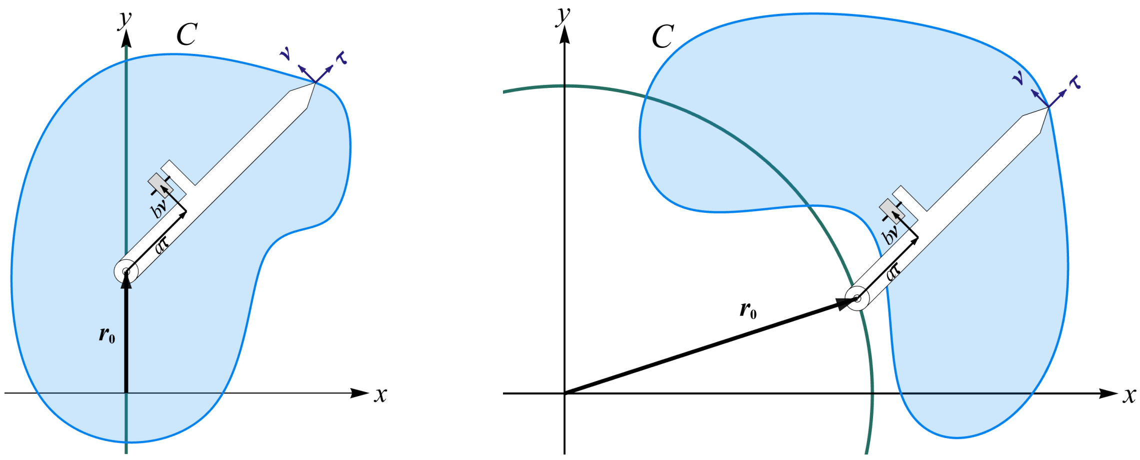
In the last section, this will be proved with Examples 4 and 5 to be introduced now.

## 3 Curves traced by wheel after class

Planimeters are mechanical instruments for measuring areas of planar regions [7]. The Amsler linear and polar planimeters are particularly simple, accurate, and more portable than others. In their simplified forms, both are rigid rod with fixed length $L$ and a wheel attached such that one end of the rod traces over $C$ while the other end, the hinge, is restricted to move on a track: a straight line (Figure 2a) for the linear version, and a circle (Figure 2b) for the polar version. The attached wheel can be at any location relative to the rod as long as its axis is parallel to the rod. It can be proved using Green's Theorem [9, 10, 6] that after the rod traces over $C$ once, if the winding number of the rod with respect to its hinge is 0, then the wheel turns multiplied by the rod length indeed equals $\text{Area}(D)$.

To ensure that the winding number is 0, we impose a limit on an angle $\phi$ between the rod and the track: $|\phi| < \pi/2$. For the linear planimeter, $\phi$ is between the rod and the upward direction of the track modeled by the $y$-axis. For the polar planimeter, $\phi$ is between the rod and the tangent vector to the counterclockwise orientated circular track centered at the origin with radius $l \geq L$. Thus chosen, if the tracing point is anywhere in the vertical strip $-L < x < L$ for the linear case, and the annular sector $l - L < r < l + L$ for the polar case, the location of the hinge is also uniquely determined. Let $\boldsymbol{r_0}$ be the position vector of the hinge, $\boldsymbol{\tau}$ the unit vector pointing from the hinge to the tracing point on the other end of the rod, and $\boldsymbol{\nu}$ the unit vector perpendicular to $\boldsymbol{\tau}$ such that the determinant $\begin{vmatrix} \boldsymbol{\tau} \\ \boldsymbol{\nu} \end{vmatrix}$ is positive. Let $\boldsymbol{r_w}$ be the position vector of the center of the measuring wheel, then $\boldsymbol{r_w} = \boldsymbol{r_0} + a\boldsymbol{\tau} + b\boldsymbol{\nu}$, where $a$ and $b$ are the offset distances as shown in Figure 2. As $\boldsymbol{r_0}$, $\boldsymbol{\tau}$, and $\boldsymbol{\nu}$ are continuously differentiable functions with respect to the components $x$ and $y$ of the position vector $\boldsymbol{r} = \langle x, y \rangle$ of the tracing point on $C$, so is $\boldsymbol{r_w}$. The total wheel measure $M$ after the tracing point goes around $C$ once is $\int_C \boldsymbol{\nu} \cdot d\boldsymbol{r_w}$, because the wheel only records turns along the direction of $\boldsymbol{\nu}$.

**Theorem 3.** *For both the linear and polar planimeters, $L \cdot M = \text{Area}(D)$ for general regions $D$.*




**(a)** *linear planimeter* **(b)** *polar planimeter*

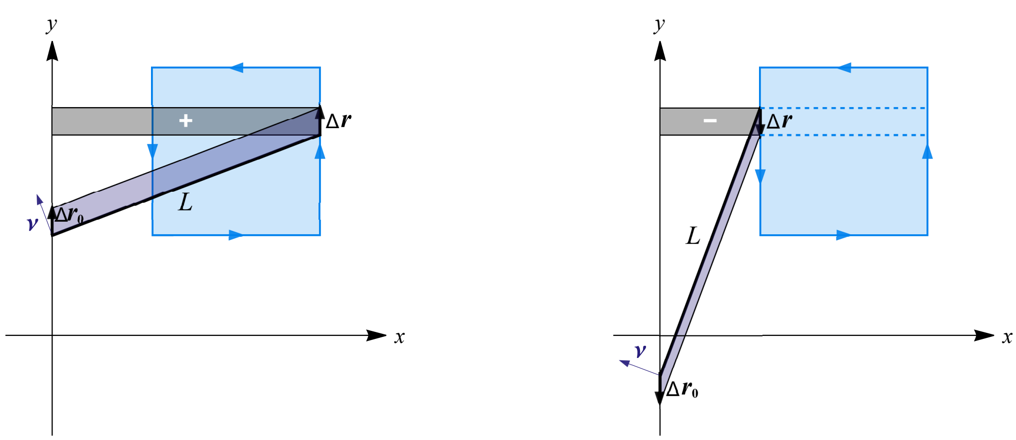
**Figure 2:** *The Amsler linear and polar planimeters.*

Before we prove Theorem 3 over elementary regions below, note that

$$L \cdot M = L \int_C \boldsymbol{\nu} \cdot d\boldsymbol{r_0} + La \int_C \boldsymbol{\nu} \cdot d\boldsymbol{\tau} + Lb \int_C \boldsymbol{\nu} \cdot d\boldsymbol{\nu}.$$

Write $\boldsymbol{\tau} = \langle \cos\theta, \sin\theta \rangle$, and $\boldsymbol{\nu} = \langle -\sin\theta, \cos\theta \rangle$, where $\theta$ is the angle measured at the hinge from the right-pointing direction counterclockwise to the rod. Then $Lb \int_{C'} \boldsymbol{\nu} \cdot d\boldsymbol{\nu} = 0$ for any segment $C'$ of $C$. Thus we can drop this integral altogether. On the other hand, $La \int_C \boldsymbol{\nu} \cdot d\boldsymbol{\tau} = La \int_C d\theta$, which is also 0 as the winding number of the rod is 0. For the linear planimeter, we will drop this integral. For the polar planimeter, we will keep this integral but let $a$ be very particular: $a = \frac{L}{2}$. So we are adding a particular 0. This choice will make our story transparent. Before that, I was quite frustrated. I got this idea when watching little B. playing baseball. Not really watching. While he did, I was scribbling down shapes and symbols in the chair, not paying much attention to the field. I got many ideas this way, which I will write about in the next years.

**Example 4. Linear Planimeter**




**(a)** **(b)**

**Figure 3:** *Elementary region of Type I for the linear planimeter.*

Let $D$ be an elementary region of Type I. Given $L \cdot M = \int_C L\boldsymbol{\nu} \cdot d\boldsymbol{r_0}$, note that $\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0}$ is the height of the slanted parallelogram with one edge $\Delta\boldsymbol{r}$ on a vertical segment of $C$, and thus $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0}$ is the signed area of this parallelogram, which is also the signed area of the rectangle with thickness $\Delta\boldsymbol{r}$. Thus, summing $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0}$ for the two $\Delta\boldsymbol{r}$ on the same height, we get the area of the horizontal strip bounded by the two dashed lines, same as that in Example 1. (See Figure 3.) On the other hand, $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0}$ add up to 0 for the two $\Delta\boldsymbol{r}$ with the same $x$ coordinate on the top and bottom sides of $C$. Therefore, $L \cdot M = \text{Area}(D)$.

**Example 5. Polar Planimeter**

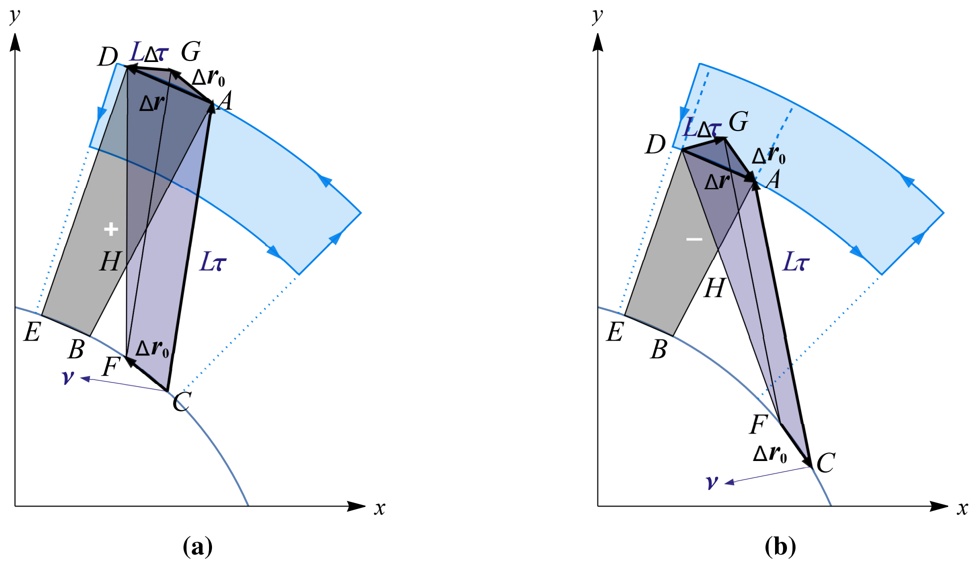
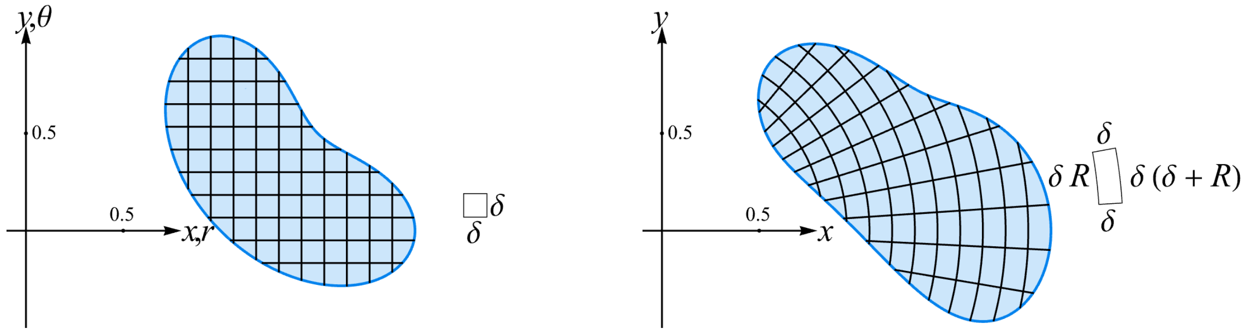



**Figure 4:** *Elementary region of Type II for the polar planimeter.*

Let $D$ be an elementary region of Type II, and we have $L \cdot M = \int_C L\boldsymbol{\nu} \cdot d\boldsymbol{r_0} + \int_C a\boldsymbol{\nu} \cdot Ld\boldsymbol{\tau}$, where $a = \frac{L}{2}$. In Figure 4, $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0}$ is the signed area of the parallelgram $CAGF$ and $\frac{L}{2}\boldsymbol{\nu} \cdot L\Delta\boldsymbol{\tau}$ is the signed area of the triangle $FGD$, which add up to the signed area of the pentagon $CAGDF$. This area is the same as that of the other pentagon $BAGDE$. To see it, first note that the triangles $CAB$ and $FDE$ are congruent and so have equal areas. As the triangles $CFB$ and $EBF$ are also congruent, the quadrilaterals $CABF$ and $FDEB$ have equal areas. After subtracting the area of $FHB$, then $CAHF$ and $BHDE$ also have qual areas. Now viewing $CAGDF$ as the union of $CAHF$ and $AGDH$, and $BAGDE$ as the union of $BHDE$ and $AGDH$ proves the claim. As $\Delta\boldsymbol{r}$ are taken on opposite curved sides of $C$ bounded between two radial lines, the two $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0} + \frac{L}{2}\boldsymbol{\nu} \cdot L\Delta\boldsymbol{\tau}$ add up to the area bounded between the dashed lines inside $D$ with some excess which goes to zero once limit is taken. If $\Delta\boldsymbol{r}$ are taken on opposite straight sides and they have the same distances to the origin, the two values of $L\boldsymbol{\nu} \cdot \Delta\boldsymbol{r_0} + \frac{L}{2}\boldsymbol{\nu} \cdot L\Delta\boldsymbol{\tau}$ cancel each other. Therefore, $L \cdot M = \text{Area}(D)$.

## 4 When the region is no longer elementary

The method used in the proof of Theorems 2 and 3 under the more general assumptions of (1) and (2), where the length of $C$ is meaningful, is classical. It was used in the proof of strong forms of Green's Theorem and Cauchy's Theorem when the boundary curve is Jordan rectifiable. Let the total length of $C$ be $\Lambda$. All line integrals in Examples 1–5 can be written in the form $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F}$ as an algebraic expression with non-vanishing denominator is continuous on $D \cup C$. For any $\epsilon > 0$, we will prove that $|\int_C \boldsymbol{F} \cdot d\boldsymbol{r} - \text{Area}(D)| < \epsilon$ to show $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \text{Area}(D)$. That $C$ has a finite length and the continuity of $\boldsymbol{F}$ are two of the three ingredients in the proof. The other is the Bliss-Hu subdividing lemma [4, 13, 8, 12, 1]. It states that for any $\delta > 0$, the




**(a)** *of Type I* **(b)** *of Type II*

**Figure 5:** *General regions and their decompositions into elementary regions.*

lines $x = m\delta$ and $y = n\delta$, $m, n \in \mathbb{Z}$ form a system of squares with edge length $\delta$ which divide $D \cup C$ into finitely many closed subregions $D_1, \cdots, D_n$ among which each of $D_1, \cdots, D_p$ is bordered by segments of $C$ in addition to edges on $x = m\delta$ and $y = n\delta$, and $D_{p+1}, \cdots, D_n$ are squares contained in the open $D$. See Figure 5a. Furthermore, $p < 4(\frac{\Lambda}{\delta} + 1)$. To see the last point, cut $C$ into contiguous half-open half closed intervals of lengths $\delta$. Then the number of such intervals is strictly less than $\frac{\Lambda}{\delta} + 1$. On the other hand, each such interval is contained in the union of four closed squares in the system arranged as the Chinese character 田. Thus, there are less than $4(\frac{\Lambda}{\delta} + 1)$ such squares covering the subregions $D_1, \cdots, D_p$ whose union contains $C$.

First, consider Examples 1– 4. We choose our $\delta$ similarly to the proofs in [4, 13, 8, 12, 1]. As $\boldsymbol{F}$ is continuous over the compact $D \cup C$, $\boldsymbol{F}$ is uniformly continuous on $D \cup C$. Thus, there is $\delta > 0$ which we can also choose to satisfy $\delta < \frac{-\Lambda + \sqrt{\Lambda^2 + \epsilon/2}}{2}$ and $\delta < 1$ such that whenever $(x_1, y_1)$ and $(x_2, y_2)$ are in the same $D_i$ contained in a closed square with edge length $\delta$ in the system, we have $|\boldsymbol{F}(x_1, y_1) - \boldsymbol{F}(x_2, y_2)| < \frac{\epsilon}{2(17\Lambda + 16)}$.

Now, apply the Bliss-Hu subdividing lemma for the above chosen $\delta$. Let $C_i$ be the boundary of $D_i$ with length $\Lambda_i$. Then we have $\int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r} = \text{Area}(D_i)$, for $i = p+1, \cdots, n$, as $D_i$ is an elementary region of Type I. Therefore,

$$\begin{aligned} |\textstyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} - \text{Area}(D)| &= |\textstyle\sum_{i=1}^n \int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r} - \sum_{i=1}^n \text{Area}(D)| \\ &= |\textstyle\sum_{i=1}^p \int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r} - \sum_{i=1}^p \text{Area}(D)| \\ &\leq \textstyle\sum_{i=1}^p |\int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r}| + \sum_{i=1}^p \text{Area}(D_i) \end{aligned}$$

Let $\boldsymbol{F_0}$ be the value of $\boldsymbol{F}$ at any chosen point in $D_i$, then $\int_{C_i} \boldsymbol{F_0} \cdot d\boldsymbol{r} = 0$ as the constant vector field $\boldsymbol{F_0}$ is conservative. Thus, $|\int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r}| = |\int_{C_i} (\boldsymbol{F} - \boldsymbol{F_0}) \cdot d\boldsymbol{r}| < \frac{\epsilon}{2(17\Lambda+16)} \Lambda_i$. So $\sum_{i=1}^p |\int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r}| < \frac{\epsilon}{2(17\Lambda+16)} \sum_{i=1}^p \Lambda_i$. Note that each $\Lambda_i$ is the sum of the length of the part on $C$ and the length of the part on a square. Thus, $\sum_{i=1}^p \Lambda_i < \Lambda + 4\delta p < \Lambda + 4\delta 4(\frac{\Lambda}{\delta} + 1) = \Lambda + 16(\Lambda + \delta)\delta < 17\Lambda + 16$ as $\delta < 1$. Therefore, $\sum_{i=1}^p |\int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r}| < \frac{\epsilon}{2}$.

On the other hand, $\sum_{i=1}^p \text{Area}(D_i) < 4(\frac{\Lambda}{\delta} + 1)\delta^2 = 4(\Lambda\delta + \delta^2) < \frac{\epsilon}{2}$, because $\delta < \frac{-\Lambda + \sqrt{\Lambda^2 + \epsilon/2}}{2}$.

Therefore, $|\int_C \boldsymbol{F} \cdot d\boldsymbol{r} - \text{Area}(D)| < \epsilon$.

For Example 5, we will decompose the domain $D$ into elementary regions of Type II as shown in Figure 5b. Consider the continuously differentiable transformation $\pi$ given by $x = r\cos\theta$, $y = r\sin\theta$ from the $r\theta$

open right half-plane to the punctured $xy$ plane $\mathbb{R}^2\backslash\{(0,0)\}$. This is a smooth covering map. Let $s$ be a section. As $D \cup C$ does not go through the origin, $D \cup C$ is in the domain of $s$. Then, for any $\delta > 0$, $s(D \cup C)$ on the $r\theta$ half-plane (Figure 5a) admits a subdivision by the lines $r = m\delta$, $\theta = n\delta$, $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, into subregions $D'_1, \cdots, D'_p$ and $D'_{p+1}, \cdots, D'_n$ as in the subdividing lemma. Let $\lambda$ be the length of $s(C)$. Then we have $p < 4(\frac{\lambda}{\delta} + 1)$.

Now map the regions $D'_i$ to $D_i$ via $\pi$. So $D_i$, $i = 1, \cdots, n$ form a subdivision of $D$ into subregions of which $D_1, \cdots, D_p$ are regions containing points of $C$ and $D_{p+1}, \cdots, D_n$ are elementary regions of Type II subtending angle $\delta$ with straight edges also measuring $\delta$. As $D \cup C$ is compact, it is enclosed in the interior of a circle with radius $R$. Then each $D_i$ has area less than $R\delta^2$, and each annular sector having intersection with $D \cup C$ has boundary length less than $\delta(2R + 2 + \delta)$. See Figure 5b.

Given any $\epsilon > 0$, as $\boldsymbol{F}$ is uniformly continuous over $D \cup C$, there is $\delta > 0$ which we also choose to satisfy $\delta < \frac{-\lambda+\sqrt{\lambda^2+\frac{\epsilon}{2R}}}{2}$ and $\delta < 1$ such that whenever $(x_1, y_1)$ and $(x_2, y_2)$ are in the same $D_i$, we have $|\boldsymbol{F}(x_1, y_1) - \boldsymbol{F}(x_2, y_2)| < \frac{\epsilon}{2(\Lambda+4(\lambda+1)(2R+3))}$. Then it can be readily checked as for Examples 1–4 that $|\int_C \boldsymbol{F} \cdot d\boldsymbol{r} - \text{Area}(D)| < \epsilon$.

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Dedicated to Tanya L. Leise, 10.27.1971–1.18.2023.

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## References


[1] Apostol, T. (1957) *Mathematical Analysis: A Modern Approach to Advanced Calculus.* First edition. Addison-Wesley.

[2] Arnold, V. I. (1989) *Mathematical Methods of Classical Mechanics*. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein. Corrected reprint of the second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York. xvi+516 pp.

[3] Baryshnikov, Y., Ghrist, R. (2018) Stokes's theorem, data, and the polar ice caps. *Amer. Math. Monthly* 125(9): 830-834.

[4] Bliss, G. A. (1934) Fundamental existence theorems. *AMS Colloquium Publication 3: The Princeton Colloquium 1909*. AMS, New York.

[5] Coffman, A., Zhang, Y. (2020) Vector fields with continuous curl but discontinuous partial derivatives. *Amer. Math. Monthly* 127(10): 911-917.

[6] Eggers, J. (2020) A closer look at the compensating polar planimeter. *College Math. J.* 51(2): 105-116.

[7] Foote, R., Sandifer, E. (2007) Area without integration: make your own planimeter. In: Shell-Gellasch, A., eds. *Hands on History: A Resource for Teaching Mathematics*. MAA Notes, pp. 71-88.

[8] Hu, K. (1935) On Cauchy's Integral Theorem. *Science Reports of the Nat. Tsing Hua Univ.*, Series A, 3: 179-183.

[9] Gatterdam, R. (1981) Classroom Notes: The planimeter as an example of Green's Theorem. *Amer. Math. Monthly* 88(9): 701-704

[10] Leise, T. (2007) As the planimeter's wheel turns: planimeter proofs for calculus class. *College Math. J.* 38(1): 24-31.

[11] Needham, T. (1997) *Visual Complex Analysis*. Oxford University Press, Oxford. xxiv+592 pp.

[12] Potts, D. H. (1951). A Note on Green's Theorem. *J. London Math. Soc.* 26(4): 302–304.

[13] Wiley, F. B., Bliss, G. A. (1914) A method of subdividing the area enclosed by a plane curve, with an application to Cauchy's theorem. *Bull. Sci. Lab. Denison Univ.* 17: 375-389.