Comps Study Guide for Linear Algebra

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This study guide was written to help you prepare for the linear algebra portion of the Comprehensive and Honors Qualifying Examination in Mathematics. It is based on the *Syllabus for the Comprehensive Examination in Linear Algebra* (*Math 271/272*) available on the Department website.

Each topic from the syllabus is accompanied by a brief discussion and examples from old exams. When reading this guide, you should focus on three things:

- *Understand the ideas.* If you study problems and solutions without understanding the underlying ideas, you will not be prepared for the exam.
- *Understand the strategy of each proof.* Most proofs in this guide are short—the hardest part is often knowing how to start. Focus on the setup step rather than falling into the trap of memorizing proofs.
- *Understand the value of scratchwork.* Sometimes scratchwork is needed to explore possible approaches to a computation or proof. That is, sometimes it is helpful to work out some details on the side by trial and error before writing up a clear, presentable solution.

The final section of the guide has some further suggestions for how to prepare for the exam.

1 Vector Spaces and Subspaces

Some basic things to be aware of, although they do not arise directly on the comps:

- The definition of a *vector space*: A set with operations called addition and scalar multiplication (by elements of $\mathbb{R}$) satisfying a certain long list of axioms.
- One of the equivalent definitions of a *subspace*: as a subset W of a vector space V such that W is a vector space in its own right, with the same addition and scalar multiplication operations.

The elements of a vector space V are called *vectors*, even if those elements are functions, matrices, or other objects. The additive identity of V is called the *zero vector*, and it is usually denoted $\mathbf{0}$ or $\mathbf{0}_V$.

Simple vector space examples. Here are some vector spaces you should know, each with standard addition and scalar multiplication operations. You don't need to memorize the notation (other than for $\mathbb{R}^n$, which you surely already know), because any such notation will be defined on the exam if it appears.

- $\mathbb{R}^n$ is the vector space of ordered n-tuples of real numbers. Sometimes denoted $\mathbf{R}^n$. Sometimes its elements are written as row vectors $(x_1, \ldots, x_n)$ and sometimes as columns. Note: $\dim(\mathbb{R}^n) = n$.
- $P_n(\mathbb{R})$ is the vector space of polynomials of degree *less than or equal to* n. Sometimes denoted P_n or P^n or $\mathcal{P}_n$ or some other such variation. Note: $\dim(P_n(\mathbb{R})) = n + 1$.
- $M_{m\times n}(\mathbb{R})$ is the vector space of $m \times n$ matrices with real entries. Sometimes denoted $\mathbf{M}_{m\times n}$ or some other such variation. Note: $\dim(M_{m\times n}(\mathbb{R})) = mn$.

There are also some infinite-dimensional vector spaces which arise occasionally. For example, $P(\mathbb{R})$ is the vector space of *all* polynomials, $F(\mathbb{R})$ is the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, and $C(\mathbb{R})$ is the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Again, don't bother memorizing the notation, which will be fully defined if it appears; just be ready to work with such spaces if they show up on the exam.

Subspace Theorem. The following Theorem is usually used to check whether a given subset is a subspace; in fact, some books use it as the definition of a subspace.

Theorem. Let V be a vector space. A subset $W \subseteq V$ is a subspace if it satisfies the following properties:

1. $W \neq \varnothing$.
2. For all $\mathbf{x}, \mathbf{y} \in W$ and all $c \in \mathbb{R}$, we have $c\mathbf{x} + \mathbf{y} \in W$.

Property 1 is usually verified by proving that $\mathbf{0}_V \in W$.
Property 2 can be replaced by two separate statements:

Closed under addition: For all $\mathbf{x}, \mathbf{y} \in W$, we have $\mathbf{x} + \mathbf{y} \in W$.

Closed under scalar multiplication: For all $\mathbf{x} \in W$ and $c \in \mathbb{R}$, we have $c\mathbf{x} \in W$.

That is, a subset W of a vector space V is a subspace *if and only if* W is nonempty, closed under addition, and closed under scalar multiplication.
Note:

- The empty set $\varnothing$ is **not** a vector space. Instead, the smallest vector space is the trivial space, $\{\mathbf{0}\}$.
- Every vector space V has two obvious subspaces: the trivial subspace $\{\mathbf{0}\} \subseteq V$, and the improper subspace $V \subseteq V$. (Obviously, the two coincide if and only if $V = \{\mathbf{0}\}$ is trivial.)

1 (March 2006) Let U and V be subspaces of a vector space W.
(a) Prove that $U \cap V$ is a subspace of W.
(b) Prove that $U + V = \{\mathbf{u} + \mathbf{v} \,:\, \mathbf{u} \in U, \mathbf{v} \in V\}$ is a subspace of W.
(c) Give an example to show that $U \cup V$ need not be a subspace of W.

Proof. (a): (Nonempty) We have $\mathbf{0}_W \in U$ and $\mathbf{0}_W \in V$, since both U and V are subspaces. Hence $\mathbf{0}_W \in U \cap V$.
(Closure) Given $\mathbf{x}, \mathbf{y} \in U \cap V$ and $c \in \mathbb{R}$, we have $c\mathbf{x} + \mathbf{y} \in U$ and $c\mathbf{x} + \mathbf{y} \in V$ since both are subspaces. Hence $c\mathbf{x} + \mathbf{y} \in U \cap V$. QED (a)

(b): (Nonempty) We have $\mathbf{0}_W \in U$ and $\mathbf{0}_W \in V$, since both are subspaces. Hence $\mathbf{0}_W + \mathbf{0}_W \in U + V$.
(Closure) Given $\mathbf{x}, \mathbf{y} \in U + V$ and $c \in \mathbb{R}$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$. Thus,

$$c\mathbf{x} + \mathbf{y} = c(\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (c\mathbf{u}_1 + \mathbf{u}_2) + (c\mathbf{v}_1 + \mathbf{v}_2) \in U + V \qquad \text{QED (b)}$$

(c): Let $W = \mathbb{R}^2$, let $U = \mathrm{Span}(\{(1,0)\})$, and let $V = \mathrm{Span}(\{(0,1)\})$. [That is, U is the x-axis, and V is the y-axis.] Then U and V are subspaces because they are each the span of a set. However, $(1,0) \in U \subseteq U \cup V$ and $(0,1) \in V \subseteq U \cup V$, but $(1,0) + (0,1) = (1,1) \notin U \cup V$. Thus, $U \cup V$ is **not** closed under addition and hence is not a subspace. QED (c)

Comment 1. In the "Nonempty" step of (b), there was no need to observe that $\mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$. Yes, that's true, and it wouldn't hurt to say it, but it would be unnecessary. We're simply trying to show that $U + V \neq \varnothing$, so all we need to do is produce an element; we aren't required to simplify that element. On the other hand, in part (c), it was important to simplify the sum $(1,0) + (0,1)$, to verify that the result is not an element of $U \cup V$.
Comment 2. In part (c), we made use of Span (to be discussed below) because it made the proof shorter, rather than verifying by hand that both U and V are subspaces. Also in part (c), there are many ways to do this. (Any choice of W with any subspaces U and V will work, as long as $U \not\subseteq V$ and $V \not\subseteq U$.) But it's generally best to pick as simple an example as you can.

Linear combinations, etc. Let V be a vector space, and let $S = \{\mathbf{v}_1 \ldots, \mathbf{v}_n\} \subseteq V$ be a finite set of vectors in V.

- A *linear combination* of elements of S is an expression $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ for some scalars $a_1, \ldots, a_n \in \mathbb{R}$.
- The *span* of S, denoted $\mathrm{Span}(S)$, is the set of all linear combinations of elements of S. That is,
$$\mathrm{Span}(S) = \{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \,|\, a_1, \ldots, a_n \in \mathbb{R}\}.$$
  - **Fact**: $\mathrm{Span}(\varnothing) = \{\mathbf{0}\}$.
  - **Fact**: $\mathrm{Span}(S)$ is a subspace of V.
  - **Fact**: If W is a subspace and $S \subseteq W$, then $\mathrm{Span}(S) \subseteq W$.
  - If $\mathrm{Span}(S) = W$, we say that S spans W.
- The set S is *linearly dependent* if there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ that are **not all zero** such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$.

  S is *linearly independent* if it is not linearly dependent. Equivalently, for any scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, we must have $a_1 = \cdots = a_n = 0$.
- The set $S \subseteq V$ is a *basis* for V if S is linearly independent and $\mathrm{Span}(S) = V$.
- The *dimension* of V is the number $\dim(V)$ of elements in a basis for V. (It is a **Theorem** that **any two bases for V have the same number of elements**.) If V has no finite basis, we say $\dim(V) = \infty$.

Side notes: Linear independence really should include an extra specification that we first ensure $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are all distinct; but this subtlety does not arise on the comps. Also, there are notions of linear combinations, span, and linear (in)dependence for an *infinite* set $S \subseteq V$, but again, these do not arise on the comps.

2 (March 2007) Suppose that V is a vector space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V$ is linearly independent. Also assume that $\mathbf{v} \in V$ is not contained in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Prove that $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

Proof. Given scalars $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $a_0\mathbf{v} + a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, we need to show that $a_0 = a_1 = \cdots = a_n = 0$.

Case 1: Suppose $a_0 = 0$. Then $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, and hence $a_1 = \cdots = a_n = 0$, because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent. Thus, $a_0 = a_1 = \cdots = a_n = 0$, as desired.

Case 2: Suppose $a_0 \neq 0$. Then $a_0\mathbf{v} = -a_1\mathbf{v}_1 - \cdots - a_n\mathbf{v}_n$, and multiplying both sides by $1/a_0$ (which *is* a real number, since $a_0 \neq 0$), we have $\mathbf{v} = -\frac{a_1}{a_0}\mathbf{v}_1 - \cdots - \frac{a_n}{a_0}\mathbf{v}_n \in \mathrm{Span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$, contradicting the hypotheses. Thus, Case 2 cannot happen, and we are done by Case 1. QED

Comment 1. Don't start a proof by regurgitating what the hypotheses mean. Instead, focus on the statement you're asked to deduce: for this problem, that the set $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent. So our first line should be to suppose we are given scalars a_i for which $a_0\mathbf{v} + a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, and our last line should be to conclude that $a_0 = a_1 = \cdots = a_n = 0$. Once you know the first and last line, *then* you can go back and try to figure out how the hypotheses could help you get there.

Comment 2. In this problem, two cases are required. We discovered that by doing some scratchwork, and realizing that the span hypothesis could only be used in one situation, and the linear independence hypothesis can be used in the other. *Scratchwork is important!*

Row reduction. Many problems in linear algebra end up requiring row reduction. You will need to know:

- How to set up a system of equations when necessary.
- The three kinds of row reduction steps. (Switching two rows, multiplying a row by a nonzero scalar, and adding a multiple of one row to another.)
- The overall strategy, leading to a matrix in *echelon form*.
- How to interpret the echelon form to solve your problem.

At the most basic level, you solve a linear system by row reducing its augmented matrix:

$$\begin{aligned} x_1 + 2x_2 \qquad + 2x_4 \qquad &= 2 \\ 2x_2 + 4x_2 + \ x_3 + 7x_4 \qquad &= 5 \\ x_1 + 2x_2 - 2x_3 - 4x_4 + x_5 &= 3 \end{aligned} \implies \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 2 \\ 2 & 4 & 1 & 7 & 0 & 5 \\ 1 & 2 & -2 & -4 & 1 & 3 \end{array}\right] \to \cdots \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right].$$

You should do this row reduction yourself (see **6** for a worked out example). The echelon matrix above has pivots [first nonzero entries of rows of an echelon form] in columns 1, 3 and 5. Thinking of the first five columns as corresponding to the variables, we see that x_2 and x_4 are free variables [since there are no pivots in these columns]. So the system of equations given by the echelon form is straightforward to solve:

$$\begin{aligned} x_1 + 2x_2 \qquad + 2x_4 \qquad &= 2 \\ x_3 + 3x_4 \qquad &= 1 \\ x_5 &= 3 \end{aligned} \implies \begin{aligned} x_1 &= 2 - 2x_2 - 2x_4 \\ x_3 &= 1 - 3x_4 \\ x_5 &= 3 \end{aligned}$$

Notice how we solve for the pivot variables x_1, x_3, x_5 in terms of the free variables. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2x_2 - 2x_4 \\ x_2 \\ 1 - 3x_4 \\ x_4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } x_2, x_4 \in \mathbb{R} \text{ are arbitrary.}$$

On the linear algebra comps, one often encounters homogeneous systems, where all the constant terms are zero. In this example, the corresponding homogeneous system is

$$\begin{aligned} x_1 + 2x_2 \qquad + 2x_4 \qquad &= 0 \\ 2x_2 + 4x_2 + \ x_3 + 7x_4 \qquad &= 0 \\ x_1 + 2x_2 - 2x_3 - 4x_4 + x_5 &= 0 \end{aligned}$$

with row reduction and general solution

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 0 \\ 2 & 4 & 1 & 7 & 0 & 0 \\ 1 & 2 & -2 & -4 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

We will say more about matrices in Section 3 of this study guide. For now, we mention two important uses of row reduction and the echelon form:

- The *nullspace* of a matrix A is the solution set of its corresponding homogeneous system of equations. The method above produces a basis of the nullspace of A, namely the set of vectors that the free variables end up multiplied by in the solution. That is, in the example above,

$$\text{the nullspace of } \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 2 & 4 & 1 & 7 & 0 \\ 1 & 2 & -2 & -4 & 1 \end{bmatrix} \text{ has basis } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

- The *column space* of a matrix A is the span of its columns. If B is the echelon form of A, then the columns of A corresponding to the columns of B with pivots form a basis of the column space. In the example above, the echelon form has pivots in columns 1, 3 and 5, so that

$$\text{the column space of } \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 2 & 4 & 1 & 7 & 0 \\ 1 & 2 & -2 & -4 & 1 \end{bmatrix} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

See **6** for a similar problem, and see Section 2 for more on nullspaces and column spaces.

Row reduction arises most obviously when you are asked to solve an specific system of equations. But it can also arise at other times, especially on problems that involve explicit vectors. Here are some examples of problems involving linear independence and span where row reduction arises naturally.

3 (March 2016) Is the following set of polynomials linearly independent? Explain your answer:

$$\{x^3 - 3x + 1, x^2 + 2x + 2, x^3 - 2x^2\}.$$

Solution/Proof. **Yes**. Call the three polynomials f, g, h.
Given scalars $a, b, c \in \mathbb{R}$ such that $af + bg + ch = 0$, we have

$$(a + c)x^3 + (b - 2c)x^2 + (-3a + 2b)x + (a + 2b) = 0,$$

i.e.,

$$a + c = 0, \quad b - 2c = 0, \quad -3a + 2b = 0, \quad a + 2b = 0.$$

Thus, (a, b, c) is a solution of the system of equations described by $\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right]$.

Row reduction [omitted here; try it yourself!] leads to the echelon form $\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

There are no free variables. The third row of the echelon form gives $c = 0$, so the second gives $b = 0$, and the third gives $a = 0$.
Since the only solution (a, b, c) to $af + bg + ch = 0$ is $a = b = c = 0$, the set $\{f, g, h\}$ is linearly independent. QED

4 (January 2015) Suppose that V is a vector space and $\mathbf{u}$ and $\mathbf{v}$ are vectors in V. Show that $\mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}) = \mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$.

Proof. ($\subseteq$): Clearly, $3\mathbf{u} + \mathbf{v} \in \mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$, and $\mathbf{u} - \mathbf{v} \in \mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$. Thus, since $\mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$ is a subspace, we have $\mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}) \subseteq \mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$.

($\supseteq$): Note that $4\mathbf{u} = (3\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v})$, so $\mathbf{u} = \frac{1}{4}(3\mathbf{u} + \mathbf{v}) + \frac{1}{4}(\mathbf{u} - \mathbf{v}) \in \mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$.
Similarly, $4\mathbf{v} = (3\mathbf{u} + \mathbf{v}) - 3(\mathbf{u} - \mathbf{v})$, so $\mathbf{v} = \frac{1}{4}(3\mathbf{u} + \mathbf{v}) - \frac{3}{4}(\mathbf{u} - \mathbf{v}) \in \mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$.
Since $\mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$ is a subspace, we have $\mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}) \supseteq \mathrm{Span}(\{\mathbf{u}, \mathbf{v}\})$. QED

Comment. The ($\supseteq$) direction required some seemingly clever choices of coefficients. How did we find them? By doing scratchwork! For example, when showing $\mathbf{u} \in \mathrm{Span}(\{3\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$, we needed to find scalars $a, b \in \mathbb{R}$ so that $a(3\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{v}) = \mathbf{u}$. This becomes the system $\left[\begin{array}{rr|r} 3 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right]$, which we solved (in scratchwork) to get $a = \frac{1}{4}$ and $b = \frac{1}{4}$.

Be sure you are comfortable with row reduction and solving linear systems. See problems **6**, **7**, **16**, **17**, **20**, **21**, **22** and **23** of this study guide for more problems that require row reduction.

Side Note: A matrix B is in echelon form if it fits the staircase pattern, where each pivot has all zeros below it and to the left of it. B is in *reduced* echelon form if, in addition, each pivot also has all zeros *above* it, and each pivot is exactly 1. Some of the echelon forms in this guide are reduced (e.g., in the example on page 4), and some are not (e.g., in Example **3**). On the comps, you may use either kind, as you prefer. On the one hand, the reduced echelon form can make it easier to write down the solution; on the other hand, if you only need the number or location of the pivots, a non-reduced echelon form suffices.

2 Linear Transformations

Let V and W be vector spaces, and let $T : V \to W$ be a function. We say T is a *linear transformation* (or a *linear map*, or simply that T is *linear*) if

$$\text{for all } \mathbf{x}, \mathbf{y} \in V \text{ and all } c \in \mathbb{R}, \text{ we have } T(c\mathbf{x} + \mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y}).$$

- The linearity property can be replaced by two separate statements:

  Respects addition: For all $\mathbf{x}, \mathbf{y} \in V$, we have $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.
  Respects scalar multiplication: For all $\mathbf{x} \in V$ and $c \in \mathbb{R}$, we have $T(c\mathbf{x}) = cT(\mathbf{x})$.

- If $U : W \to X$ is another linear transformation, then the composition $U \circ T : V \to X$ is also linear. The composition $U \circ T$ is often denoted simply UT.

- Don't confuse proving "T is linear" with "W is a subspace." They are similar but *not* the same. (In particular, proving W is a subspace *also* requires proving it's nonempty.)

- Sometimes we drop the parentheses and write $T\mathbf{x}$ instead of $T(\mathbf{x})$.

- When T is linear, the following identity can be very useful:

  $$T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n).$$

- If $A \in M_{m\times n}(\mathbb{R})$ is an $m \times n$ matrix, then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$ is linear.

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there is a matrix $A \in M_{m\times n}(\mathbb{R})$ so that T is given by $T(\mathbf{x}) = A\mathbf{x}$.

Kernel, image, nullity, and rank. Let $T : V \to W$ be a linear transformation.
The *kernel* or *nullspace* of T is

$$\{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\} \subseteq V.$$

It is usually denoted either $\mathrm{Ker}(T)$ or $N(T)$, or sometimes $\mathrm{nullspace}(T)$. Its dimension $\dim(\mathrm{Ker}(T))$ is called the *nullity* of T, sometimes denoted $\mathrm{nullity}(T)$.

The *image* or *range* of T is

$$\{T(\mathbf{v}) \mid \mathbf{v} \in V\} = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ s.t. } T(\mathbf{v}) = \mathbf{w}\} \subseteq W.$$

It is usually denoted either $\mathrm{Im}(T)$ or $R(T)$, or sometimes $\mathrm{range}(T)$. Its dimension $\dim(\mathrm{Im}(T))$ is called the *rank* of T, sometimes denotes $\mathrm{rank}(T)$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by the matrix A, the range of T is sometimes called the *column space* of A, because it is precisely the span of the columns of A.

- **Fact**: $N(T)$ is a subspace of V, and $R(T)$ is a subspace of W.

- **Fact**: T is one-to-one if and only if $N(T) = \{\mathbf{0}_V\}$, i.e., iff $N(T)$ is as small as possible, i.e., iff $\mathrm{nullity}(T) = 0$.

  (Recall $f : X \to Y$ is *one-to-one* means for all $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, we have $x_1 = x_2$.)

- **Fact**: T is onto if and only if $R(T) = W$, i.e., iff $R(T)$ is as big as possible. If $\dim(W) < \infty$, this is equivalent to saying $\mathrm{rank}(T) = \dim(W)$.

  (Recall $f : X \to Y$ is *onto* means for all $y \in Y$, there exists $x \in X$ with $f(x) = y$.)

- **Fact**: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a matrix A, then

  $$N(T) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \quad \text{and} \quad R(T) = \text{column space of } A.$$

  Furthermore, after finding an echelon form of A via row reduction, $\mathrm{rank}(T)$ is the number of columns *with* pivots (since the corresponding columns form a basis of $R(T)$), and $\mathrm{nullity}(T)$ is the number of columns *without* pivots (since the vectors that give the general solution form a basis of $N(T)$).

- Given a linear map $T : V \to W$, know how to use row reduction to find bases for both $N(T)$ and $R(T)$. Also be able to decide whether or not T is one-to-one, and whether or not T is onto.

Some proof problems combine the notions of linear maps, span, linear independence, one-to-one, and/or onto, so be prepared for problems like the following.

5 (March 2013) Suppose that V and W are vector spaces, $T : V \to W$ is a linear transformation, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Prove that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ spans W and T is one-to-one, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V.

Proof. Given $\mathbf{v} \in V$, we need to show $\mathbf{v} \in \text{Span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$.
Since $T(\mathbf{v}) \in W = \text{Span}(\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\})$, there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$T(\mathbf{v}) = a_1 T(\mathbf{v}_1) + \cdots + a_n T(\mathbf{v}_n) = T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n).$$

Since T is one-to-one, we have $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \in \text{Span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$. QED

Comment 1. As always in a proof, start by focusing on the goal. We need to show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V; so ask yourself what the first and last line of such a proof should be.

Comment 2. If you know T is linear, then there's a good chance that you will have to use the fact that $T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n)$. So be ready to do so.

Here is a problem about finding bases for the column space and null space of a particular matrix.

6 (March 2014) Find [bases for] the column space and the null space (or kernel) for the matrix $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \end{bmatrix}$.

Solution. We do row reduction:

$$\begin{matrix} \\ -\text{R1} \\ -2\text{R1} \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \end{bmatrix} \to \begin{matrix} +2\text{R2} \\ \\ -\text{R2} \end{matrix} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to \times(-1) \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This echelon form matrix has pivots in columns 1 and 3. So those columns of the original matrix A are a basis for the column space. That is,

$$(\text{column space of } A) = \text{Span}\{(1, 1, 2), (2, 1, 3)\}.$$

For the kernel, the free variables are x_2, x_4, and the echelon form gives the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \implies \text{Ker}(A) = \text{Span}\{(-2, 1, 0, 0), (-1, 0, -1, 1)\}.$$

Comment 1. When you do row reduction, be sure to indicate what operations you are doing at each step. Do this both to avoid errors and to avoid losing excess points if you *do* make an error.

Comment 2. Don't do two row reduction operations in one step **unless** those two operations commute. For example, you can add multiples of row 1 to both rows 2 and 3 in one step, but you can't add row 1 to row 2 **and** multiply row 2 by -1 in the same step.

Sometimes you may be asked to find a basis for some subspace when there is no obvious linear map or matrix. In that case, usually the subspace is actually the kernel or image of some linear map or matrix which will make itself apparent once you starting playing with it. Here is an example of such a problem.

7 (March 2009) Let V be the vector space of polynomials with real coefficients and of degree at most 3, and let $W = \{f \in V \mid f(0) = f''(0) \text{ and } f'(1) = 0\}$.

(a) Prove that W is a subspace of V.

(b) Find a basis for W.

(a): *Proof.* (Nonempty) Let $f = 0 \in V$. Then $f' = 0$ and $f'' = 0$, so $f(0) = 0 = f''(0)$ and $f'(0) = 0$. That is, $f \in W$. So $W \neq \varnothing$.

(Closure) Given $f, g \in W$ and $c \in \mathbb{R}$, we have

$$(cf + g)(0) = cf(0) + g(0) = cf''(0) + g''(0) = (cf + g)''(0)$$

and

$$(cf + g)'(1) = cf'(1) + g'(1) = c \cdot 0 + 0 = 0,$$

and hence $cf + g \in W$. QED (a)

(b): *Solution.* Write an element f of $V = P_3(\mathbb{R})$ as $f = a + bx + cx^2 + dx^3$. Then $f' = b + 2cx + 3dx^2$, and $f'' = 2c + 6dx$. So $f(0) = a$, $f''(0) = 2c$, and $f'(1) = b + 2c + 3d$. Hence,

$$W = \{a + bx + cx^2 + dx^3 \in V \mid a = 2c \text{ and } b + 2c + 3d = 0\}.$$

Thus, we are solving the system $a - 2c = 0$ and $b + 2c + 3d = 0$ for $a, b, c, d \in \mathbb{R}$. That is, we need the kernel of $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, which is already in echelon form. Since there are pivots only in the columns for a and b, the free variables are c and d.

Setting $c = 1$ and $d = 0$ gives $a = 2$ and $b = -2$, yielding $x^2 - 2x + 2$.

Setting $c = 0$ and $d = 1$ gives $a = 0$ and $b = -3$, yielding $x^3 - 3x$.

So $\{x^2 - 2x + 2, x^3 - 3x\}$ is a basis for W.

Comment. In the moment, it can be easy to forget that we're dealing with elements of V, which are polynomials. So when you finish the problem, make sure your answer actually fits. In particular, the answer to (b) is *not* $\{(2, -1, 1, 0), (0, -3, 0, 1)\}$, but rather $\{x^2 - 2x + 2, x^3 - 3x\}$.

Some facts about dimension.

- **Fact**: If $X \subseteq V$ is a subspace, then $\dim(X) \leq \dim(V)$.
  Moreover, if $\dim(V) < \infty$, then $\dim(X) = \dim(V)$ if and only if $X = V$.
- **Fact**: Let V be a vector space with $\dim(V) = n$, and let $S \subseteq V$ be a set of m distinct vectors in V.
  - If $m < n$, then S **cannot** span V.
  - If $m > n$, then S **cannot** be linearly independent.

Here are two useful facts that might be called "two-out-of-three" theorems:

- **Theorem**: Let V be a vector space, and let $S \subseteq V$ be a set of n distinct vectors in V. If any two of the following conditions hold, then all three hold (and S is a basis for V):
  1. S is linearly independent.
  2. S spans V.
  3. $\dim(V) = n$.
- **Theorem**: Let $T : V \to W$ be a linear transformation, and suppose that at least one of V, W is finite-dimensional. If any two of the following conditions hold, then all three hold:
  1. T is one-to-one.
  2. T is onto.
  3. $\dim(V) = \dim(W)$.

  In this case, T is *invertible*. (More on this later; see page 13 of this Guide.)

Here is a very important fact, sometimes called the *Rank-Nullity Theorem* or the *Dimension Theorem*.

- **Theorem**: Let $T : V \to W$ be a linear transformation. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

8 (January 2012) Let C be a 3×5 real-valued matrix. Answer the following questions about C and briefly justify your answers:

(a) Can the columns of C be linearly independent?

(b) Does the equation $C\mathbf{x} = \mathbf{0}$ have a unique solution with $\mathbf{x} \in \mathbb{R}^5$?

(c) Assume that the span of the columns of C is all of $\mathbb{R}^3$. Can you determine the nullity (=dimension of the null space or kernel) of C?

Answers. (a): **No**, the columns of C must be linearly dependent. The columns of C are five vectors in $\mathbb{R}^3$. Since $\dim(\mathbb{R}^3) = 3 < 5$, any set of five vectors in $\mathbb{R}^3$ is linearly dependent.

(b): **No**, the equation $C\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Since the range $R(C)$ is a subspace of $\mathbb{R}^3$, we have $\operatorname{rank}(C) \leq 3$. Since multiplication-by-C is a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$, the Rank-Nullity Theorem yields

$$\operatorname{nullity}(C) = \dim(\mathbb{R}^5) - \operatorname{rank}(C) \geq 5 - 3 = 2.$$

Thus, the solution set $N(C)$ to $C\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^5$ of dimension at least 2, so it has infinitely many elements.

(c): **Yes, we must have** $\operatorname{nullity}(C) = 2$. The rank of C is by definition $\operatorname{rank}(C) = \dim(R(C))$. Since the range $R(C)$ is the span of the columns of C, the hypothesis says that $R(C) = \mathbb{R}^3$, and hence $\operatorname{rank}(C) = 3$. Thus, the Rank-Nullity Theorem yields

$$\operatorname{nullity}(C) = \dim(\mathbb{R}^5) - \operatorname{rank}(C) = 5 - 3 = 2.$$

Comment. An alternate way to do this problem is to say that C has some echelon form B, with some number of pivots between 0 and 3. (Max of 3, since there are 3 rows.) Then consider what the columns and rows with or without pivots say about each of parts (a), (b), (c).

9 (January 2009) Let A be an $n \times n$ matrix with real entries such that $A^2 = 0$.

(a) Prove that the column space of A is contained in the nullspace of A. (Recall that the nullspace, or kernel, is the set of vectors $\mathbf{v} \in \mathbb{R}^n$ satisfying $A\mathbf{v} = \mathbf{0}$.)

(b) Prove that $\operatorname{rank}(A) \leq \operatorname{nullity}(A)$. (Recall that the rank is the dimension of the column space and the nullity is the dimension of the nullspace.)

(c) Prove that $\operatorname{nullity}(A) \geq \frac{1}{2}n$.

Proof (a): Write $R(A)$ for the column space of A and $N(A)$ for the nullspace.
We must show $R(A) \subseteq N(A)$. Given $\mathbf{w} \in R(A)$, there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{w} = A\mathbf{v}$. Thus,

$$A\mathbf{w} = A(A\mathbf{v}) = (A^2)\mathbf{v} = 0\mathbf{v} = \mathbf{0},$$

and hence $\mathbf{w} \in N(A)$. QED (a)

(b): Since $R(A) \subseteq N(A)$ by part (a), we have

$$\operatorname{rank}(A) = \dim(R(A)) \leq \dim(N(A)) = \operatorname{nullity}(A).$$ QED (b)

(c): The Rank-Nullity Theorem for A gives

$$n = \dim(\mathbb{R}^n) = \operatorname{rank}(A) + \operatorname{nullity}(A) \leq 2\operatorname{nullity}(A),$$

where the inequality is from part (b). Dividing both sides by 2, we have $\operatorname{nullity}(A) \geq \dfrac{1}{2}n$. QED (c)

10 (March 2011) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ are linear transformations, and for all $\mathbf{v} \in \mathbb{R}^2$, $(T \circ U)(\mathbf{v}) = \mathbf{v}$.

(a) Prove that T is onto.

(b) What is the nullity of T? Is T one-to-one? Justify your answers.

Proof. (a): Given $\mathbf{v} \in \mathbb{R}^2$, let $\mathbf{w} = U(\mathbf{v}) \in \mathbb{R}^3$. Then

$$T(\mathbf{w}) = T(U(\mathbf{v})) = (T \circ U)(\mathbf{v}) = \mathbf{v}.$$

QED (a)

(b): We have $\text{nullity}(T) = 1$. **No**, T is not one-to-one.
To see this, note by part (a) that the range $R(T)$ is all of $\mathbb{R}^2$, because T is onto. Thus, the rank of T is $\text{rank}(T) = \dim(R(T)) = 2$. So by the Rank-Nullity Theorem,

$$\text{nullity}(T) = \dim(\mathbb{R}^3) - \text{rank}(T) = 3 - 2 = 1.$$

Since the nullspace $N(T)$ has dimension $\text{nullity}(T) = 1 > 0$, T is **not** one-to-one. QED (b)

11 (February 2007) Let V and W be vector spaces over $\mathbb{R}$ and let $T : V \to W$ be a linear map. Assume that V has a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans W.

(a) Prove that T is onto.

(b) If $\dim V = \dim W$, what else can you conclude about T? Explain your reasoning.

Proof. (a): Given $\mathbf{w} \in W$, there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $\mathbf{w} = a_1 T(\mathbf{v}_1) + \cdots + a_n T(\mathbf{v}_n)$, since $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans W. Thus,

$$\mathbf{w} = T\big(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n\big)$$

lies in the image of T, as desired. QED (a)

(b): **T is one-to-one and onto (and hence invertible)**. We know from part (a) that T is onto, and we know from the hypothesis of (b) that $\dim(W) = \dim(V) = n < \infty$. Thus, by a two-out-of-three theorem, T must also be one-to-one. QED (b)

12 (January 2012) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose T is one-to-one (injective). Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is also a basis for V.

Proof. First, we claim that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent. Given scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1 T(\mathbf{v}_1) + \cdots + a_n T(\mathbf{v}_n) = \mathbf{0}$, we have

$$T\big(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n\big) = \mathbf{0}.$$

Since T is one-to-one, it has trivial kernel, so $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$. Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, it follows that $a_1 = \cdots = a_n = 0$, proving our claim.

Now $\dim(V) = n$ (because by hypothesis, it has a basis consisting of n elements), and by our claim, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent set of n elements in V. Thus, by a two-out-of-three theorem, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ also spans V and hence is a basis for V. QED

3 Matrices

Know:

- an $m \times n$ matrix is an m-row, n-column grid of real numbers.
- how to add two $m \times n$ matrices.
- how to multiply AB, where $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$.
- matrix addition is commutative and associative.
- matrix multiplication is associative *but not commutative.*
- the $m \times n$ matrix of zeros (denoted 0 or O) is the additive identity: $A + O = O + A = A$.
- the $n \times n$ identity matrix I, sometimes denoted I_n, has 1's down the diagonal and 0's everywhere else. It satisfies $AI = A$ and $IB = B$ (for A is $m \times n$ and B is $n \times p$).
- the distributive laws $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
- How to convert a system of linear equations to a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

In addition, if A is a *square* matrix (i.e., $n \times n$), know

- A is a *diagonal matrix* if every entry not on the diagonal is 0.
- The *transpose* of A, denoted A^t or A^T, is the $n \times n$ matrix formed by reflecting A across the diagonal.
- The *determinant* of A, denoted $\det(A)$, is a real number given by a more complicated formula. Know:
  - How to compute $\det(A)$ for a given $n \times n$ matrix, for any $n \geq 1$.
  - The fact that $\det(AB) = \det(A)\det(B)$.
  - The fact that $\det(A) \neq 0$ if and only if A is invertible (see later).
  - An alternate notation for a determinant is to put vertical lines instead of brackets around the entries of the matrix. For example, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ means the same thing as $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, namely $ad - bc$.

Determinants don't usually arise directly on comps exams, but you will need determinants to find eigenvalues.

Matrix of a linear transformation. Let V and W be finite-dimensional vector spaces with bases $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively, and let $T : V \to W$ be a linear transformation.

Find real numbers a_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n$, by computing $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ and expressing each as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$. That is,

$$\begin{aligned} T(\mathbf{v}_1) &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\ T(\mathbf{v}_2) &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\ &\vdots \\ T(\mathbf{v}_n) &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m. \end{aligned}$$

Then, **turning the grid of coefficients sideways**, the *$m \times n$ matrix of T with respect to α and β* is

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Coordinate vectors. Let V be a finite-dimensional vector space, and let $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. For any $\mathbf{v} \in V$, the *coordinate vector* of $\mathbf{v}$ with respect to α is the n-entry column vector

$$[\mathbf{v}]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \quad \text{where } a_1, \dots, a_n \in \mathbb{R} \text{ are the unique scalars such that } \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

(The term "coordinate vector" and the notation $[\mathbf{v}]_\alpha$ and can vary from textbook to textbook. If a comps problem needs to refer to coordinate vectors, the problem itself will define the notation and terminology.)

- **Key Fact 1**: For any $\mathbf{v} \in V$, the coordinate vectors $[\mathbf{v}]_\alpha \in \mathbb{R}^n$ and $[T(\mathbf{v})]_\beta \in \mathbb{R}^m$ are related via the matrix $[T]_\alpha^\beta \in M_{m\times n}(\mathbb{R})$ by the equation
$$[T]_\alpha^\beta [\mathbf{v}]_\alpha = [T(\mathbf{v})]_\beta.$$
- **Key Fact 2**: If X is another vector space, with basis $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_p\}$, and if $U : W \to X$ is linear, then the $m \times n$ matrix $[T]_\alpha^\beta$, the $n \times p$ matrix $[U]_\beta^\gamma$, and the $m \times p$ matrix $[UT]_\alpha^\gamma$ are related by matrix multiplication:
$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

13 (January 2009) Let $M_{2\times 2}$ be the vector space of 2×2 matrices with real entries. It is a fact that $\dim M_{2\times 2} = 4$, and that $\{e_1, e_2, e_3, e_4\}$ is a basis for $M_{2\times 2}$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the function $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = A \begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix}$.

(a) Prove that T is linear.

(b) Find the matrix for T with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

Solution. For ease of notation, define $C = \begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix}$ and $\alpha = \{e_1, e_2, e_3, e_4\}$.

(a): Given $A, B \in M_{2\times 2}$ and $c \in \mathbb{R}$, we have

$$T(A + cB) = (A + cB)C = AC + (cB)C = AC + c(BC) = T(A) + cT(B) \qquad \text{QED (a)}$$

(b): We compute

$$T(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 0 & 0 \end{pmatrix} = -1e_1 + 7e_2 + 0e_3 + 0e_4,$$
$$T(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3e_1 + 4e_2 + 0e_3 + 0e_4,$$
$$T(e_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 7 \end{pmatrix} = 0e_1 + 0e_2 - 1e_3 + 7e_4,$$
$$T(e_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0e_1 + 0e_2 + 3e_3 + 4e_4$$

Putting these coefficients down columns, the matrix is $[T]_\alpha^\alpha = \begin{bmatrix} -1 & 3 & 0 & 0 \\ 7 & 4 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 7 & 4 \end{bmatrix}$.

Comment. Make sure you understand why the matrix for T in part (b) is 4×4, not 2×2.

Here are two problems you should try yourself. See also Example **17** later.

> **14** (March 2013) Let P_2 be the vector space consisting of all polynomials of degree at most 2, and let $B = \{1, x+1, x^2+x+1\}$, which is a basis for P_2. Let $T : P_2 \to P_2$ be the linear transformation defined by the equation $T(f) = f - f'$. Find the matrix of T with respect to B.
>
> ANSWER: $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

> **15** (January 2014) Suppose $T : V \to V$ is a linear transformation, $\boldsymbol{\mathcal{B}} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for V, and the matrix representation of T with respect to $\boldsymbol{\mathcal{B}}$ is
>
> $$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & -3 \\ -1 & 19 & 0 \end{bmatrix}.$$
>
> Determine $T(2\mathbf{b}_1 + 4\mathbf{b}_3)$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. ANSWER: $24\mathbf{b}_1 + 2\mathbf{b}_2 - 2\mathbf{b}_3$

Invertible maps. A linear transformation $T : V \to W$ is *invertible* if it has an *inverse*, i.e., if there is another linear map $T^{-1} : W \to V$ such that $T(T^{-1}(\mathbf{w})) = \mathbf{w}$ for all $\mathbf{w} \in W$, and $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in V$. Know that **the following are equivalent**:

- T is invertible (i.e., T has an inverse $T^{-1} : W \to V$).
- T is one-to-one and onto.

In addition, recall that one of the earlier two-out-of-three theorems (see page 8 of this Guide) says the following: if either $\dim(V) < \infty$ or $\dim(W) < \infty$ (or both), then T being invertible is equivalent to each of the following statements:

- $\dim(V) = \dim(W)$ and T is one-to-one.
- $\dim(V) = \dim(W)$ and T is onto.

Facts: Suppose $T : V \to W$ is invertible. Then:

- Its inverse T^{-1} is unique.
- Its inverse T^{-1} is also invertible, and $(T^{-1})^{-1} = T$.
- If $U : W \to X$ is also invertible, then $UT : V \to X$ is invertible, and $(UT)^{-1} = T^{-1}U^{-1}$.

Invertible matrices. A square matrix $A \in M_{n\times n}(\mathbb{R})$ is *invertible* if it has an *inverse*, i.e., if there is another matrix $B \in M_{n\times n}(\mathbb{R})$ such that $AB = I$ and $BA = I$. Know that **the following are equivalent**:

- A is invertible
- The columns of A are linearly independent.
- The rows of A are linearly independent.
- The columns of A together span $\mathbb{R}^n$.
- The rows of A together span $\mathbb{R}^n$.
- The columnspace (i.e., range or image) of A is all of $\mathbb{R}^n$.
- The nullspace (i.e., kernel) of A is $\{\mathbf{0}\}$.
- $\text{rank}(A) = n$.
- $\text{nullity}(A) = 0$.
- $\det(A) \neq 0$.
- $\lambda = 0$ is **not** an eigenvalue of A.

Warning: the equivalencies above are only for **square** matrices. Nonsquare matrices are *never* invertible.

Facts: Suppose $A \in M_{n\times n}(\mathbb{R})$ is invertible. Then:

- Its inverse A^{-1} is unique.
- Its inverse A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.
- If $B \in M_{n\times n}(\mathbb{R})$ is invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Know how to compute inverses:

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- For 3×3 and larger matrices, use the Gauss-Jordan method (row reduction), as in the next example.

16 (February 2006) Compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{pmatrix}.$$

Check your answer by matrix multiplication.

Solution. We do row reduction:

$$\begin{matrix}\downarrow \\ \uparrow \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \to -2\text{R1} \\ -4\text{R1} \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \to \\ -\text{R2} \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & -4 & 1 \end{array}\right]$$

$$\to -\text{R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right]$$

So $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -1 & -2 & 1 \end{pmatrix}$. Now we check by multiplication:

$$AA^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Comment: We only need to check either $AA^{-1} = I$ **or** $A^{-1}A = I$. Because A is square, if one is true, then the other is automatically true. (Can you use the facts about invertible matrices to see why?)

Change of basis. Let V be a finite-dimensional vector space, and let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ both be bases for V. The *change-of-basis matrix* from α-coordinates to β-coordinates is the $n \times n$ matrix $[I]_\alpha^\beta$, where $I : V \to V$ is the identity map $I(\mathbf{v}) = \mathbf{v}$. That is,

$$[I]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \text{where} \quad \begin{array}{rcl} \mathbf{v}_1 & = & a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{n1}\mathbf{w}_n \\ \mathbf{v}_2 & = & a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{n2}\mathbf{w}_n \\ & \vdots & \\ \mathbf{v}_n & = & a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{nn}\mathbf{w}_n \end{array}$$

As before, note that the grid of coefficients is flipped sideways to form the matrix.

- For any $\mathbf{v} \in V$, recall that $[\mathbf{v}]_\alpha \in \mathbb{R}^n$ is the n-entry column vector of α-coordinates for $\mathbf{v}$. We can compute the β-coordinate vector for $\mathbf{v}$ via the formula $[\mathbf{v}]_\beta = [I]_\alpha^\beta[\mathbf{v}]_\alpha$.
- The change of coordinates matrix to change coordinates the other way is the inverse: $[I]_\beta^\alpha = ([I]_\alpha^\beta)^{-1}$.

Change of basis matrices are often used when we can easily find out the matrix of some linear map $T : V \to W$ for one basis but we actually want the matrix of T for another basis, as in the following example.

17 (January 2012) Let $T : P_2 \to \mathbb{R}^2$, where $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, be defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

You may assume that T is linear.

(a) Find bases of the null space (kernel) and range of T.

(b) Find the matrix representation of T with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Solution. Let $\alpha = \{1, t, t^2\}$, which is the standard basis for P_2, and let $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$, which is the standard basis for $\mathbb{R}^2$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have

$$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 1\mathbf{e}_2, \quad T(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\mathbf{e}_1 + 2\mathbf{e}_2, \quad T(t^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\mathbf{e}_1 + 4\mathbf{e}_2,$$

and so the associated matrix is $[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. We now answer the questions:

(a): Row reduction gives echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$, which has pivots in the first and second columns. So the corresponding columns $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ of the original matrix form a basis for the range of T. For the null space, name variables a, b, c for the three columns; so c is free. The echelon form gives $b = -3c$ and so $a = -b - c = 2c$. That is, $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of the matrix. But we want the null space of T, and the coordinate vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ gives the polynomial $2 - 3t + t^2 \in P_2(T)$. So $\{2 - 3t + t^2\}$ is a basis for the null space of T.

(b): Let $\gamma = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. We want the matrix $[T]_\alpha^\gamma$, and we already have $[T]_\alpha^\beta$. Then the change-of-basis matrix from γ to standard (β) coordinates on $\mathbb{R}^2$ is

$$[I]_\gamma^\beta = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

i.e., just the vectors of γ down the columns. [Make sure you understand why!] But we want the inverse $[I]_\beta^\gamma$, which by the 2×2 inverse formula is

$$[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So the desired matrix for T is

$$[T]_\alpha^\gamma = [T]_\alpha^\beta [I]_\beta^\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}.$$

Comment. An alternate way to do part (b) would be to work out by hand that

$$T(1) = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad T(t) = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad T(t^2) = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and then put these coefficients down columns to obtain the matrix above.

Here are two problems you should try yourself where change-of-basis matrices may be useful. Try each two ways: using change-of-basis matrices, and working things out as described in the Comment for Example **17**.

18 (March 2012) Let $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, and suppose that $T : \mathbb{R}^2 \to P_2$ is a linear transformation which satisfies

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 1 - 2t \quad \text{and} \quad T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) = t + 2t^2.$$

(a) Find $T\left(\begin{bmatrix}8\\-4\end{bmatrix}\right)$. ANSWER: $-1 + 5t + 6t^2$.

(b) Is T one-to-one? ANSWER: Yes.

(c) Is T onto? ANSWER: No.

19 (January 2015) Let V be the vector space of polynomials of degree at most 2, and let $\boldsymbol{\mathcal{B}} = \{1, x+1, x^2+x+1\}$, which is a basis for V. Suppose that $T : V \to V$ is a linear transformation, and the matrix of T relative to $\boldsymbol{\mathcal{B}}$ is

$$\begin{bmatrix}1 & 2 & 3\\3 & 0 & -1\\2 & 5 & 1\end{bmatrix}.$$

Find $T(3x^2 + x + 2)$. ANSWER: $1 - 5x - 5x^2$.

4 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix, let $\lambda \in \mathbb{R}$ be a scalar, and let $\mathbf{v} \in \mathbb{R}^n$. We say that $\mathbf{v}$ is an *eigenvector* of A with *eigenvalue* λ if $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$.

- To say "λ is an eigenvalue of A" means there exists $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- To say "$\mathbf{v}$ is an eigenvector of A" means that $\mathbf{v} \neq \mathbf{0}$ and there exists $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- An eigenvalue λ **can** be 0.
- An eigenvector $\mathbf{v}$ **cannot** be $\mathbf{0}$. (By definition.)

Side note: One can also define eigenvalues and eigenvectors for more general linear maps $T : V \to V$, as well as to allow complex numbers, but neither of these topics appears on the comps exam.

Computations. The *characteristic polynomial* of A is $\det(A - \lambda I)$; that is, subtract the variable λ from each diagonal entry and take the determinant. The result is a polynomial of degree n in the variable λ.

- **Fact**: The roots of the characteristic polynomial of A are *precisely* the eigenvalues of A.
- The number of times that a given scalar λ shows up as a root of the characteristic polynomial is called the *algebraic multiplicity* of λ, or sometimes simply the *multiplicity* of λ.
- The *eigenspace* E_λ of an eigenvalue λ is the nullspace $N(A - \lambda I)$ of the matrix $A - \lambda I$. It consists of all the eigenvectors with eigenvalue λ, along with $\mathbf{0}$, which is not an eigenvector.
- The eigenspace dimension $\dim(E_\lambda)$ is sometimes called the *geometric multiplicity* of λ.
- **Fact**: For any eigenvalue λ of A, we have

$$1 \leq (\text{geometric multiplicity of } \lambda) \leq (\text{algebraic multiplicity of } \lambda).$$

Thus, to find the eigenvalues and eigenvectors of a matrix A:

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find all roots of the characteristic polynomial; these are the eigenvalues.
3. For each eigenvalue λ, use row reduction to find a basis for the eigenspace $E_\lambda = N(A - \lambda I)$.

Diagonalization. We say an $n \times n$ matrix A is *diagonalizable* if there is an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. **The following are equivalent**:

- A is diagonalizable.
- There is a *basis* for $\mathbb{R}^n$ consisting of eigenvectors of A.
- All roots of the characteristic polynomial of A are real, and for each such root λ,

$$(\text{geometric multiplicity of } \lambda) = (\text{algebraic multiplicity of } \lambda).$$

In that case, the matrix P consists of the n linearly independent eigenvectors down the columns, and the diagonal matrix D has the eigenvalues along the diagonal, *in the same order*.

Fact: If A has n *distinct* real eigenvalues, then A is certainly diagonalizable. (However, if A has repeated eigenvalues, it **may or may not** be diagonalizable.)

20 (January 2015) Let

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

(a) Find all eigenvalues of A.
(b) Find, if possible, an invertible matrix P such that $P^{-1}AP$ is diagonal, or show that no such matrix exists.

Solution. (a): The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 4-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2-5\lambda+4-4) = -\lambda(\lambda-5)^2,$$

so the eigenvalues are $0, 5, 5$. (Or if you prefer, 0 and 5, where 5 has algebraic multiplicity 2.)

(b): Since $\lambda = 5$ has algebraic multiplicity 2, we focus on it first. We have $A - 5I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$.

Doing a quick row reduction (details omitted here) leads to the echelon form $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which has 2 pivots and hence nullity $3 - 2 = 1$. That is, the nullspace of $A - 5I$ has dimension only 1, which is strictly less than the algebraic multiplicity of $\lambda = 5$.
Since there is an eigenvalue whose eigenspace has dimension strictly less than its algebraic multiplicity, the matrix A is **not diagonalizable**.
That is, **there does not exist** an invertible matrix P such that $P^{-1}AP$ is diagonal.

Comment: When you're trying to decide whether or not A is diagonalizable, hone in on any eigenvalues of algebraic multiplicity at least 2. There's no sense wasting your time finding the eigenvectors $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ for $\lambda = 0$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ for $\lambda = 5$ if it turns out — as it does here — that the matrix isn't diagonalizable.

21 (March 2010) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ for the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Solution. First, we find the eigenvalues of A. The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} \\ &= (1-t)(-5-t)(1-t) - 27 - 27 + 9(1-t) + 9(1-t) + 9(5+t) \\ &= (1-t)(-5-t)(1-t) + 9(-6+2-2t+5+t) = (1-t)(-5-t)(1-t) + 9(1-t) \\ &= (1-t)[(t^2+4t-5)+9] = (1-t)(t^2+4t+4) = -(t-1)(t+2)^2. \end{aligned}$$

So the eigenvalues are $1, -2, -2$.

For the eigenvectors, since $\lambda = -2$ has algebraic multiplicity 2, we focus on it first. We have $A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$. A quick row reduction (omitted here) gives echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which has one pivot and hence nullity $3 - 1 = 2$, same as the algebraic multiplicity. From the echelon form, we see that the eigenspace $E_{-2} = N(A + 2I)$ has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

For $\lambda = 1$, we have $A - (1)I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$. A quick row reduction (omitted here) gives echelon form $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and so the eigenspace $E_1 = N(A + I)$ has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Since each eigenvalue has eigenspace of the same dimension as its algebraic multiplicity, the matrix A is diagonalizable, with $P^{-1}AP = D$, where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comment 1: This problem doesn't explicitly ask you to find eigenvalues or eigenvectors. You have to recognize that finding P and D as requested means finding eigenvalues and eigenvectors.

Comment 2: We do not need to compute the inverse P^{-1}. Knowing that we have a **basis** of eigenvectors is enough to guarantee that $P^{-1}AP = D$. That said, make sure the order of eigenvalues along the diagonal of P corresponds to the order of eigenvectors in the columns of P.

Warning: Don't mix up the two adjectives "diagonalizable" and "invertible". They have nothing to do with one another. For example:

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is both diagonalizable and invertible.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is diagonalizable but not invertible.

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is neither invertible nor diagonalizable.

You should try confirming those facts about all four of the above 2×2 matrices.

Here are two diagonalization problems you should try yourself.

22 (March 2008) Let

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

(a) Find all eigenvalues of A. ANSWER: $\lambda = 0, 1, 1$.

(b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, or show that there is no such matrix P. ONE ANSWER: $P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

23 (March 2009) Let M be the matrix

$$M = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of M. ANSWER: $\lambda = 2, 3, 3$.

(b) Is M diagonalizable? Why or why not? PARTIAL ANSWER: No.

Note on **22**: there is more than one possible correct answer for part (b). For example, we could permute the three columns of P any way we choose. We could also replace the column $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ by any nonzero scalar multiple, and we could replace the other two columns $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by any other basis for the (2-dimensional) eigenspace of $\lambda = 1$. For any of these changes, the columns would still be a linearly independent set of eigenvectors, making $P^{-1}AP$ still a diagonal matrix.

Note on **23**, the "Answer" to part (b) doesn't include the very important answer to "Why?" The full answer is very similar to part (b) of **20** above. In particular, a computation shows that the eigenvalue $\lambda = 3$ has algebraic multiplicity 2, but geometric multiplicity only 1.

5 Preparing for the Linear Algebra Exam

Now that you have finished reading the content part of the Study Guide, what should you do next to prepare for the linear algebra exam? The key thing to keep in mind is that

You need an active knowledge of linear algebra.

Here are a some suggestions to help you achieve this.

Read the Study Guide Actively. There are several places where the Study Guide asks you to do a problem yourself. Do so. For those problems, the Study Guide gives you the final answer so you can check your work. However, on the exam, we grade *all* of your work, not just the final answer.

Read Your Notes and Your Linear Algebra Book. In many places in the Study Guide, we say "Know. . . ," without stating the facts precisely. This is deliberate, since we want you to refer to your notes and your linear algebra book when studying for the exam. For example, this Study Guide doesn't explain the strategy of row reduction or how to compute determinants (although some examples appear in the worked problems). You are expected to review how to do that yourself.

Not everything covered in your linear algebra course is part of the exam. For example, orthogonality and the Gram-Schmidt process are important topics from linear algebra, but they are not covered on the comps exam. This Study Guide and the *Syllabus for the Comprehensive Examination in Linear Algebra* (*Math 271/272*) list the topics that you need to know.

Know Basic Results and Definitions. Keep in mind that knowing the precise statements of definitions and basic theorems is essential. The adjective "precise" is important here. For example, if a problem asks you to define an eigenvector of a matrix A, writing just

$$A\mathbf{v} = \lambda\mathbf{v}$$

will not get full credit. You need to state the whole definition: $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of A if

$$\mathbf{v} \neq \mathbf{0} \quad \text{and there exists } \lambda \in \mathbb{R} \text{ such that} \quad A\mathbf{v} = \lambda\mathbf{v}.$$

Study Old Exams and Solutions. The Department website has a collection of old Comprehensive Exams, many with solutions. It is very important to do practice problems. This is one of the key ways to acquire an active knowledge of linear algebra. However, there are two dangers to be aware of when using old exams and solutions:

- Thinking that the exams tell you what to study. Every topic on the *Syllabus* and in this Study Guide is fair game for an exam question.
- Reading the solutions. This is passive. To get an active knowledge of the material, do problems from the old exams yourself, and *then* check the solutions. The more you can do this, the better.

Work Together, Ask Questions, and Get Help. Studying with your fellow math majors can help. You can learn a lot from each other. Faculty are delighted to help. Don't hesitate to ask us questions and show us your solutions so we can give you feedback. The QCenter has excellent people who have helped many students in the past prepare for the Comprehensive Exam.

Start Now. Properly preparing for the linear algebra exam will take longer than you think. Start now.