Poisson Maps between Character Varieties

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1. Introduction

- Let $\Sigma_1$ and $\Sigma_2$ be two surfaces (possibly with boundary).
- Let $G$ be a reductive Lie group (or a compact Lie group).
- Define
  \[ X(\Sigma_1) = \text{Hom}(\pi_1, G)/G \]
  (and similarly for $\Sigma_2$), where $\pi_1$ is the fundamental group of $\Sigma_1$ and $\pi_2$ is the fundamental group of $\Sigma_2$. Here $G$ acts by conjugation.
- The spaces $X(\Sigma_j)$ are Poisson manifolds. (If the boundary of $\Sigma_j$ is empty, then $X(\Sigma_j)$ is symplectic.)
- Suppose $\Sigma_1 \subset \Sigma_2$. Let $f : \Sigma_1 \to \Sigma_2$ be the inclusion map.
- Then $f$ induces a morphism
  \[ \Phi : X(\Sigma_2) \to X(\Sigma_1), \]
  where
  \[ \Phi = f_*. \]
  Here we are thinking of
  \[ X(\Sigma_1) = \text{Hom}(\pi_1, G)/G. \]
• Strictly speaking $f_\ast : \text{Hom}(\pi_2, G) \to \text{Hom}(\pi_1, G)$ but this map is equivariant with respect to conjugation by an element of $G$.

• Sometimes we will denote a surface of genus $g$ with $n$ boundary components by $\Sigma_{n,g}$.

• We will denote $X(\Sigma_{n,g})$ by $X_{n,g}$.

• We will show that the map $\Phi$ is Poisson.
3. Poisson structures

- In local complex coordinates $z_i$, the Poisson bivector on the character variety is written
  \[ \sum_{i,j} a_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}. \]
  Here $a_{i,j}$ is a function of the $z_i$.

- Suppose $\alpha, \beta$ are based loops in $\Sigma_1$, giving rise to elements $[\alpha], [\beta]$ in $\pi_1(\Sigma)$. WLOG these based loops intersect in transverse double points.

- Let $\alpha \cap \beta$ denote the set of double point intersections.

- Let $\epsilon(\pi, \alpha, \beta)$ be the intersection number at $p \in \alpha \cap \beta$ and let $\alpha_p$ denote the curve $\alpha$ based at $p$.

- Let $\mathcal{R} : \pi \to G$ be a representation.

- Then define $f_\alpha(\mathcal{R}) = f(\mathcal{R}(\alpha))$, where $f : G \to \mathbb{C}$. 
• Let $A \in G$. Define $F(A)$ (an element of the Lie algebra of $G$) by

$$< F(A), X > = \frac{d}{dt}|_{t=0} f \left( (\exp tX)A \right)$$

where $< \cdot, \cdot >$ is an Ad -invariant inner product on the Lie algebra of $G$ and $X$ is an element of the Lie algebra of $G$.

• Here for $G = U(n)$ and $G = SU(n)$ and $f(A) = \text{Re}(\text{Trace}A)$ we have $F(A) = \frac{1}{2}(A - A^{-1})$. It can be shown that this is an element of the Lie algebra of $G$.

• Denote the fundamental group of the surface by $\pi$.

• When $\mathcal{R}$ is a homomorphism from $\pi$ to $G$, the Poisson bracket is defined by

$$\{f_\alpha(\mathcal{R}), g_\beta(\mathcal{R})\} = \sum_p \epsilon(p, \alpha, \beta) \ < F(\mathcal{R}(\alpha)), G(\mathcal{R}(\beta)) > .$$

Recall that $\epsilon(p, \alpha, \beta)$ is the intersection number of the 1-cycles $\alpha$ and $\beta$ at the intersection point $p$.

• Here we sum over points $p$ where $\alpha$ and $\beta$ intersect.
4. Maps between surfaces

• Suppose $q : \Sigma_1 \rightarrow \Sigma_2$ is a continuous map. Then there is an induced homomorphism of fundamental groups and a continuous map from $\text{Hom}(\pi_1(\Sigma_2), G)$ to $\text{Hom}(\pi_1(\Sigma_1), G)$, which descends to character varieties. This gives rise to a map $q^* : X(\Sigma_2) \rightarrow X(\Sigma_1)$.

**Theorem** [Goldman 1984, 1986]: *Let $q : \Sigma_1 \rightarrow \Sigma_2$ be a continuous map of compact orientable surfaces that preserves transversality of based loops and preserves double points.*

Then the induced map of coordinate rings $q^*$ from coordinate rings $\mathbb{C}[X(\Sigma_1)]$ to $\mathbb{C}[X(\Sigma_2)]$ is a morphism of Poisson algebras if $q$ preserves orientation and an anti-Poisson morphism if $q$ reverses orientations.

• The Poisson bracket on $X(\Sigma)$ is given by Lawton 2009 ($n > 0$), Goldman 1986 ($n = 0, g \geq 2$) and Sikora 2014 ($n = 0, g = 1$). For $n = 1, g = 1$ the Poisson bi-vector was computed (for $SL(2, \mathbb{C})$) by Goldman (2006) and (for $SL(3, \mathbb{C})$) by Lawton (2009).

• For $SL(2, \mathbb{C})$, Goldman (2006) also found the Poisson bi-vector for $g = 0, n = 4$ and $g = 1, n = 2$. 
5. Construction of Poisson structure

- Now assume the surface is an open smooth manifold (instead of a manifold with boundary).

- Assume $G$ is compact or reductive (the complexification of a compact Lie group).

- Fix a base point. A homomorphism $\mathcal{R}$ from $\pi$ to $G$ is called reductive if the Zariski closure of its image in $G$ is a reductive subgroup. (This condition is always satisfied if $G$ is compact.)

- Let $G$ be a Lie group. Then $X_{n,g}$ is the quotient (by conjugation) of the space of reductive homomorphisms of the fundamental group of a surface $\Sigma_{n,g}$ with genus $g$ and $n$ boundary components into $G$.

- A $G$-connection on $\Sigma_1$ (resp. $\Sigma_2$) is a smooth principal $G$-bundle $E_G$ on the surface $\Sigma_1$ (resp. $\Sigma_2$) equipped with a connection.

- Let $E$ be a vector bundle over $\Sigma_1$ (resp. $\Sigma_2$) associated to the principal bundle $E_G$ through the adjoint representation.

- Poincaré duality tells us that the tangent space to the character variety $X_{n,g}$ is (as usual) the compactly supported first cohomology $H^1_c(\Sigma, E)$. 
There is a natural homomorphism from $T^*X(\Sigma, G)$ to $TX(\Sigma, G)$ (or equivalently a 2-form on $X(\Sigma)$).

This is given by taking the inner product of a 1-form on $\Sigma$ with a compactly supported 1-form on $\Sigma$ to form a compactly supported 2-form on $\Sigma$, which is then integrated over $\Sigma$.

Denote this homomorphism by $\Theta : T^*X_{n,g} \rightarrow TX_{n,g}$, or equivalently $\Theta \in \Omega^2 X_{n,g}$.

If $\Sigma$ is a compact oriented surface, the 2-form $\Theta$ coincides with the symplectic structure constructed by Atiyah-Bott and Goldman.
Suppose $\Sigma_1$ is embedded as a connected open subset of $\Sigma_2$. The restriction map from $\Sigma_2$ to $\Sigma_1$ gives a map

$$\Phi : X_{n_2, g_2} \to X_{n_1, g_1}.$$ 

- We have maps
  
  $$\beta : H^1(\Sigma_2, E) \to H^1(\Sigma_1, E)$$

  (from restriction)

  $$\gamma : H^1_c(\Sigma_1, E) \to H^1_c(\Sigma_2, E)$$

  (compact support).

- The map $\beta$ is the same as the map $d\Phi$. This map results from pulling back 1-forms under the inclusion map from $\Sigma_1$ to $\Sigma_2$.

- The map $\gamma$ coincides with $(d\Phi)^*$. This map results from pushing forward from $\Sigma_1$ to $\Sigma_2$ under the inclusion map.
Theorem: The map $\Phi$ is Poisson.

• A map $F : A \to B$ is Poisson (for Poisson manifolds $A$ and $B$) if and only if

$$dF \circ \Theta_A \circ (dF')^* \cong \Theta_B$$

where $\Theta_A : T^*_x A \to T_x A$ is the Poisson structure (similarly for $B$).

• Let

$$\phi : \Sigma_1 \to \Sigma_2$$

be a (possibly ramified) covering map. Define a map $\Psi : X(\Sigma_2) \to X(\Sigma_1)$ that is given by pushing forward the fundamental group under $\phi$ from $\Sigma_1$ to $\Sigma_2$.

Theorem: The map $\Psi$ is Poisson.

The proof uses that $d\Psi$ sends a cohomology class to its pullback under $\phi$.

The map dual to $d\Phi$ sends a compactly supported 1-form to the sum of its pullback over inverse image points under the map $\phi$. This form is also compactly supported.
6. Capping

Let $\Sigma_1$ be a 2-manifold with boundary. Let $\Sigma_2$ be a 2-manifold obtained by capping the boundary components as described below. We obtain a map $\Phi : X(\Sigma_2) \to X(\Sigma_1)$ as described earlier.

(a) Capping with disks: $\Phi$ is an injection. The image is the union of a family of symplectic leaves.

(b) Capping with a cylinder (gluing the two boundary components of the cylinder
to two components of the boundary of the surface).

This produces a surface with two less boundary components and genus one higher.

The two character varieties have the same dimension.
(c) Capping with $k$-holed sphere:

The resulting surface has $k$ less boundary components, and genus $k - 1$ more.

The map $\Phi$ is not surjective.
(d) Capping one boundary component with a genus 1 curve:
If $G$ is semisimple, then the map $\Phi$ is surjective.

(e) Capping with an $n$-punctured genus 1 curve:

**Proposition:** The map $\Phi$ is surjective and Poisson.
7. Gluing via symplectic quotients.

Two collections of spaces whose quotients are character varieties:

(a) $q$-Hamiltonian spaces (Alekseev-Malkin-Meinrenken 1998):

- Let $D(G)$ be $G \times G$, with a $G \times G$-action. This space is called the double. Let $D(G)$ also be $G \times G$, with a $G$-action. This space is called the internal fusion of the double.

- The action of $G \times G$ on the double $D(G)$ is by

\[(g_1, g_2) : (a, b) \mapsto (g_1 a g_2^{-1}, g_2 b g_1^{-1})\].

- The 2-form on the double is

\[\frac{1}{2} (a^* \theta, b^* \bar{\theta}) + \frac{1}{2} (a^* \bar{\theta}, b^* \theta)\].
• The internal fusion $\mathbf{D}(G)$ is just $G \times G$ with the $G$ action

$$g : (a, b) \mapsto (\text{Ad}_g(a), \text{Ad}_g(b)).$$

• The 2-form on the internal fusion $\mathbf{D}(G)$ is

$$\omega = \frac{1}{2}(a^* \theta, b^* \bar{\theta}) + \frac{1}{2}(a^* \bar{\theta}, b^* \theta) + \frac{1}{2} \left((ab)^* \theta, (a^{-1}b^{-1})^* \bar{\theta}\right).$$

• Here $\theta$ is the left Maurer-Cartan form and $\bar{\theta}$ is the right Maurer-Cartan form. Here $\theta$ is often written as $\theta = a^{-1} da$ if $a \in G$. 
• The $q$-Hamiltonian space $D(G)^r \times (D(G))^g$ is $G^{2(g+r)}$ with the $q$-Hamiltonian action of $(z_0, \ldots, z_r) \in G^{r+1}$ given by

\begin{align*}
a_i &\mapsto \text{Ad}_{z_0} a_i \\
b_i &\mapsto \text{Ad}_{z_0} b_i \\
u_j &\mapsto z_0 u_j (z_j)^{-1} \\
v_j &\mapsto z_j v_j (z_j)^{-1}
\end{align*}

($j = 1, \ldots, r; \ i = 1, \ldots, g$)

• This space is equipped with a 2-form (playing the role of symplectic form) and moment maps

\begin{align*}
\mu_j(a, b, u, v) &= (v_j)^{-1} \quad (j = 1, \ldots, r) \\
\mu_0(a, b, u, v) &= \text{Ad}_{u_1}(v_1) \ldots \text{Ad}_{u_r}(v_r)[a_1, b_1] \ldots [a_g, b_g].
\end{align*}

• Here we have used the notation $a, b$ to refer to the tuple $(a_1, \ldots, a_g, b_1, \ldots, b_g)$.

• Similarly the notation $u, v$ refers to the tuple $(u_1, \ldots, u_r, v_1, \ldots, v_r)$.

• $[a, b]$ denotes $aba^{-1}b^{-1}$ for $a, b \in H$. 
• To form a Poisson manifold by gluing two boundary components of a connected surface, we set the values of the moment maps corresponding to those two components to be equal and then take the quotient by the diagonal action of the group.

• To form a Poisson manifold by gluing boundary components of two different surfaces, we set the moment maps on the two \( q \)-Hamiltonian spaces to be equal and take the quotient by the diagonal action of the group.

• This procedure can be iterated. If all boundary components are glued together, we recover a symplectic manifold.
References


