Poisson Maps between Character Varieties

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1. Introduction

- Let Σ_1 and Σ_2 be two surfaces (possibly with boundary).
- Let G be a reductive Lie group (or a compact Lie group).
- Define

$$X(\Sigma_1) = \text{Hom}(\pi_1, G)/G$$

(and similarly for Σ_2), where π_1 is the fundamental group of Σ_1 and π_2 is the fundamental group of Σ_2. Here G acts by conjugation.

- The spaces $X(\Sigma_j)$ are Poisson manifolds. (If the boundary of Σ_j is empty, then $X(\Sigma_j)$ is symplectic.)
- Suppose $\Sigma_1 \subset \Sigma_2$. Let $f : \Sigma_1 \to \Sigma_2$ be the inclusion map.
- Then f induces a morphism

$$\Phi : X(\Sigma_2) \to X(\Sigma_1),$$

where

$$\Phi = f_*.$$

Here we are thinking of

$$X(\Sigma_1) = \text{Hom}(\pi_1, G)/G.$$

- Strictly speaking $f_* : \mathrm{Hom}(\pi_2, G) \to \mathrm{Hom}(\pi_1, G)$ but this map is equivariant with respect to conjugation by an element of G.
- Sometimes we will denote a surface of genus g with n boundary components by $\Sigma_{n,g}$.
- We will denote $X(\Sigma_{n,g})$ by $X_{n,g}$.
- We will show that the map Φ is Poisson.

3. Poisson structures

• In local complex coordinates z_i, the Poisson bivector on the character variety is written

$$\sum_{i,j} a_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

Here $a_{i,j}$ is a function of the the z_i.

• Suppose α, β are based loops in Σ_1, giving rise to elements $[\alpha]$, $[\beta]$ in $\pi_1(\Sigma)$. WLOG these based loops intersect in transverse double points.

• Let $\alpha \cap \beta$ denote the set of double point intersections.

• Let $\epsilon(\pi, \alpha, \beta)$ be the intersection number at $p \in \alpha \cap \beta$ and let α_p denote the curve α based at p.

• Let $\mathcal{R} : \pi \to G$ be a representation.

• Then define $f_\alpha(\mathcal{R}) = f(\mathcal{R}(\alpha))$, where $f : G \to \mathbf{C}$.

- Let $A \in G$. Define $F(A)$ (an element of the Lie algebra of G) by

$$< F(A), X >= \frac{d}{dt}|_{t=0} f\Big((\exp tX)A\Big)$$

where $< \cdot, \cdot >$ is an Ad -invariant inner product on the Lie algebra of G and X is an element of the Lie algebra of G.

- Here for $G = U(n)$ and $G = SU(n)$ and $f(A) = \mathrm{Re}(\mathrm{TraceA})$ we have $F(A) = \frac{1}{2}(A - A^{-1})$. It can be shown that this is an element of the Lie algebra of G.

- Denote the fundamental group of the surface by π.

- When $\mathcal{R}$ is a homomorphism from π to G, the Poisson bracket is defined by

$$\{f_\alpha(\mathcal{R}), g_\beta(\mathcal{R})\} = \sum_p \epsilon(p, \alpha, \beta) < F(\mathcal{R}(\alpha)), G(\mathcal{R}(\beta)) > .$$

Recall that $\epsilon(p, \alpha, \beta)$ is the intersection number of the 1-cycles α and β at the intersection point p.

- Here we sum over points p where α and β intersect.

4. Maps between surfaces

• Suppose $q : \Sigma_1 \to \Sigma_2$ is a continuous map. Then there is an induced homomorphism of fundamental groups and a continuous map from $\mathrm{Hom}(\pi_1(\Sigma_2), G)$ to $\mathrm{Hom}(\pi_1(\Sigma_1), G)$, which descends to character varieties. This gives rise to a map $q_* : X(\Sigma_2) \to X(\Sigma_1)$.

Theorem [Goldman 1984, 1986]: *Let $q : \Sigma_1 \to \Sigma_2$ be a continuous map of compact orientable surfaces that preserves transversality of based loops and preserves double points.*

Then the induced map of coordinate rings q^* from coordinate rings $\mathbf{C}[X(\Sigma_1)]$ to $\mathbf{C}[X(\Sigma_2)]$ is a morphism of Poisson algebras if q preserves orientation and an anti-Poisson morphism if q reverses orientations.

• The Poisson bracket on $X(\Sigma)$ is given by Lawton 2009 ($n > 0$), Goldman 1986 ($n = 0, g \geq 2$) and Sikora 2014 ($n = 0, g = 1$). For $n = 1, g = 1$ the Poisson bi-vector was computed (for $SL(2, C)$) by Goldman (2006) and (for $SL(3, C)$) by Lawton (2009).

• For $SL(2, C)$, Goldman (2006) also found the Poisson bi-vector for $g = 0, n = 4$ and $g = 1, n = 2$.

5. Construction of Poisson structure

• Now assume the surface is an open smooth manifold (instead of a manifold with boundary).

• Assume G is compact or reductive (the complexification of a compact Lie group).

• Fix a base point. A homomorphism $\mathcal{R}$ from π to G is called reductive if the Zariski closure of its image in G is a reductive subgroup. (This condition is always satisfied if G is compact.)

• Let G be a Lie group. Then $X_{n,g}$ is the quotient (by conjugation) of the space of reductive homomorphisms of the fundamental group of a surface $\Sigma_{n,g}$ with genus g and n boundary components into G.

• A G-connection on Σ_1 (resp. Σ_2) is a smooth principal G-bundle E_G on the surface Σ_1 (resp. Σ_2) equipped with a connection.

• Let E be a vector bundle over Σ_1 (resp. Σ_2) associated to the principal bundle E_G through the adjoint representation.

• Poincaré duality tells us that the tangent space to the character variety $X_{n,g}$ is (as usual) the compactly supported first cohomology $H^1_c(\Sigma, E)$.

- There is a natural homomorphism from $T^*X(\Sigma, G)$ to $TX(\Sigma, G)$ (or equivalently a 2-form on $X(\Sigma)$).

- This is given by taking the inner product of a 1-form on Σ with a compactly supported 1-form on Σ to form a compactly supported 2-form on Σ, which is then integrated over Σ.

- Denote this homomorphism by $\Theta : T^*X_{n,g} \to TX_{n,g}$, or equivalently $\Theta \in \Omega^2 X_{n,g}$.

- If Σ is a compact oriented surface, the 2-form Θ coincides with the symplectic structure constructed by Atiyah-Bott and Goldman.

Suppose Σ_1 is embedded as a connected open subset of Σ_2. The restriction map from Σ_2 to Σ_1 gives a map

$$\Phi : X_{n_2,g_2} \to X_{n_1,g_1}.$$

- We have maps

$$\beta : H^1(\Sigma_2, E) \to H^1(\Sigma_1, E)$$

(from restriction)

$$\gamma : H^1_c(\Sigma_1, E) \to H^1_c(\Sigma_2, E)$$

(compact support).

- The map β is the same as the map $d\Phi$. This map results from pulling back 1-forms under the inclusion map from Σ_1 to Σ_2.

- The map γ coincides with $(d\Phi)^*$. This map results from pushing forward from Σ_1 to Σ_2 under the inclusion map.

Theorem: The map Φ is Poisson.

- A map $F : A \to B$ is Poisson (for Poisson manifolds A and B) if and only if

$$dF \circ \Theta_A \circ (dF)^* \cong \Theta_B$$

where $\Theta_A : T_x^* A \to T_x A$ is the Poisson structure (similarly for B).

- Let

$$\phi : \Sigma_1 \to \Sigma_2$$

be a (possibly ramified) covering map). Define a map $\Psi : X(\Sigma_2) \to X(\Sigma_1)$ that is given by pushing forward the fundamental group under ϕ from Σ_1 to Σ_2.

Theorem: The map Ψ is Poisson.

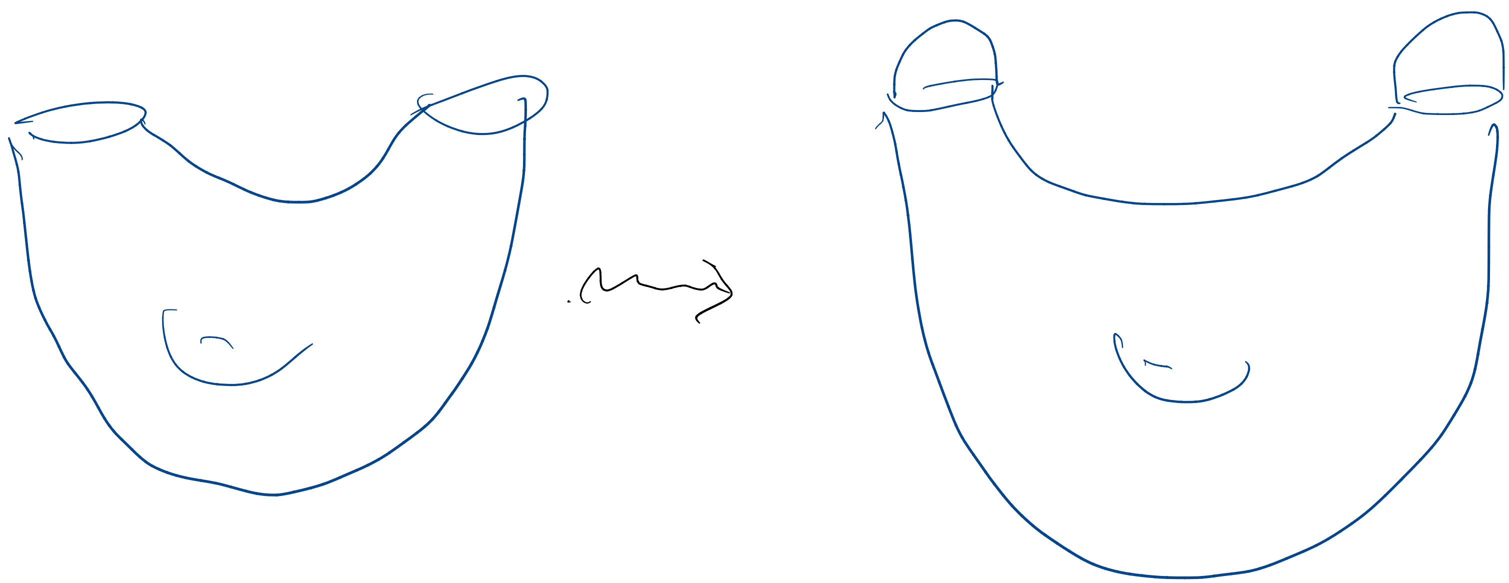
The proof uses that $d\Psi$ sends a cohomology class to its pullback under ϕ.

The map dual to $d\Phi$ sends a compactly supported 1-form to the sum of its pullback over inverse image points under the map ϕ. This form is also compactly supported.

6. Capping

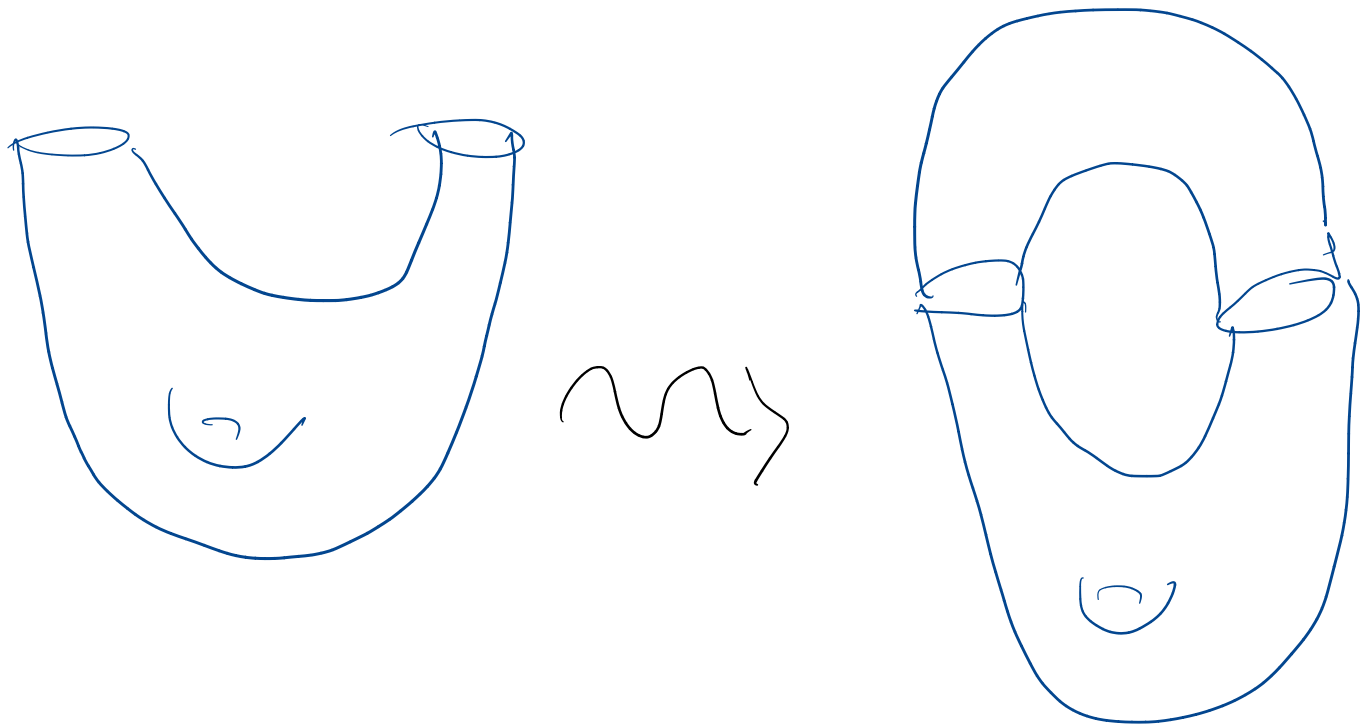
Let Σ_1 be a 2-manifold with boundary. Let Σ_2 be a 2-manifold obtained by capping the boundary components as described below. We obtain a map $\Phi : X(\Sigma_2) \to X(\Sigma_1)$ as described earlier.

(a) Capping with disks: Φ is an injection. The image is the union of a family of symplectic leaves.



(b) Capping with a cylinder (gluing the two boundary components of the cylinder to two components of the boundary of the surface).

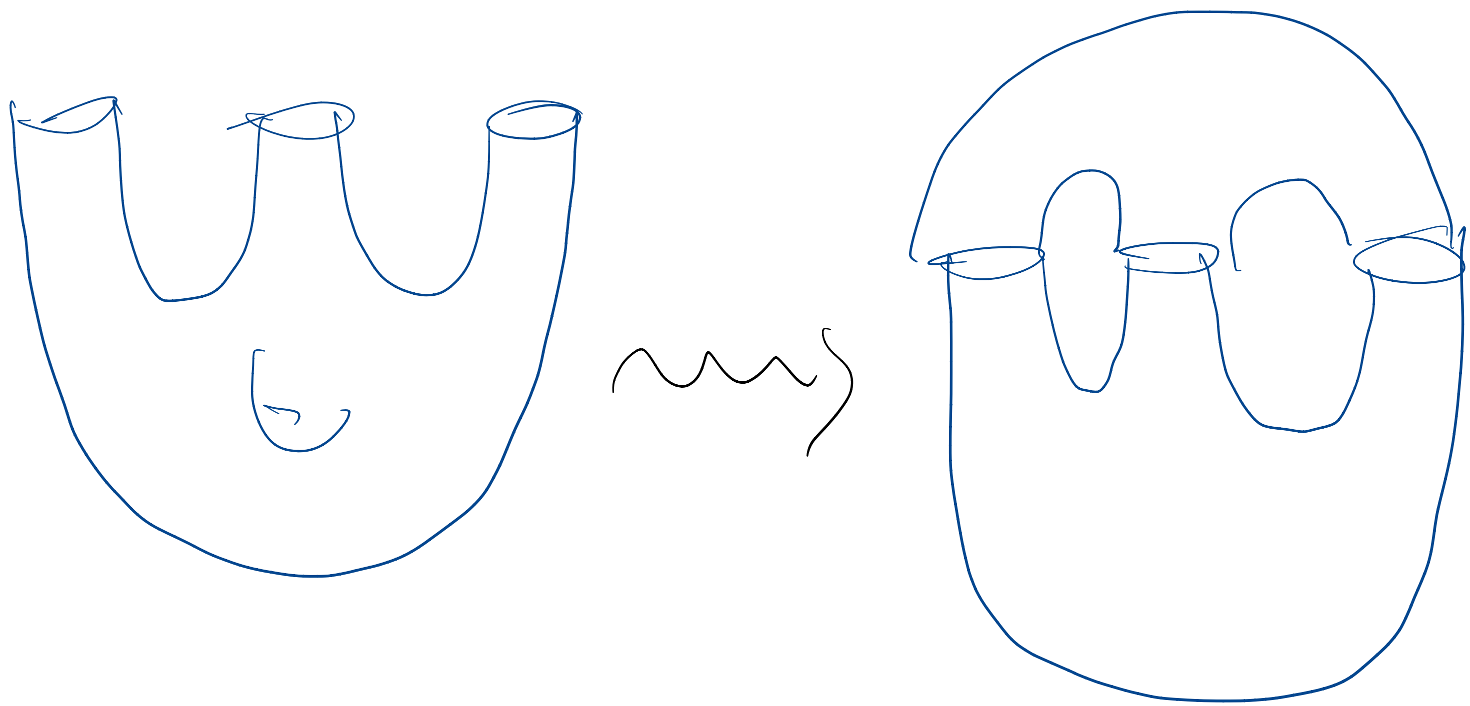
This produces a surface with two less boundary components and genus one higher.



The two character varieties have the same dimension.

(c) Capping with k-holed sphere:

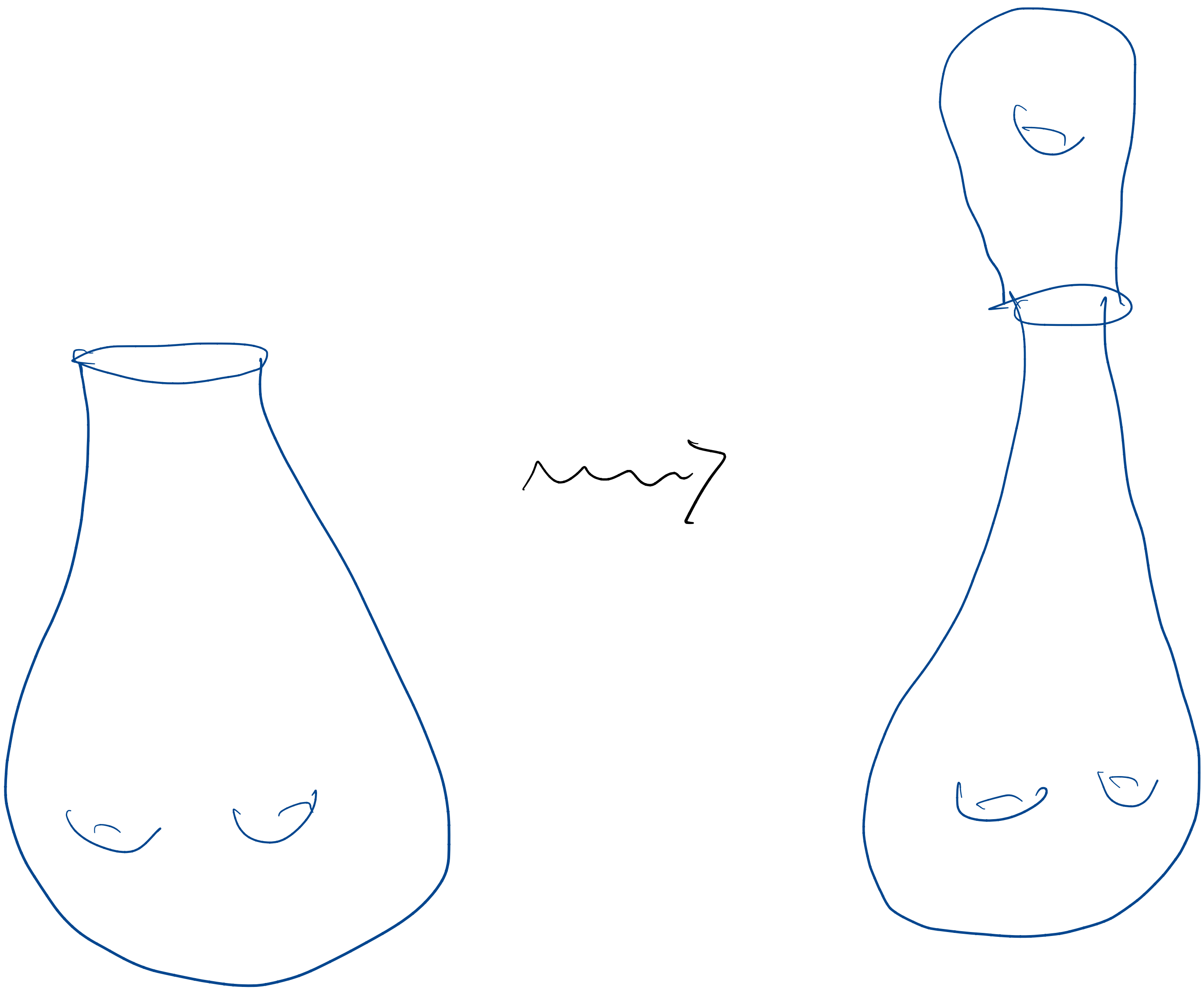
The resulting surface has k less boundary components, and genus $k-1$ more.



$(k=3)$

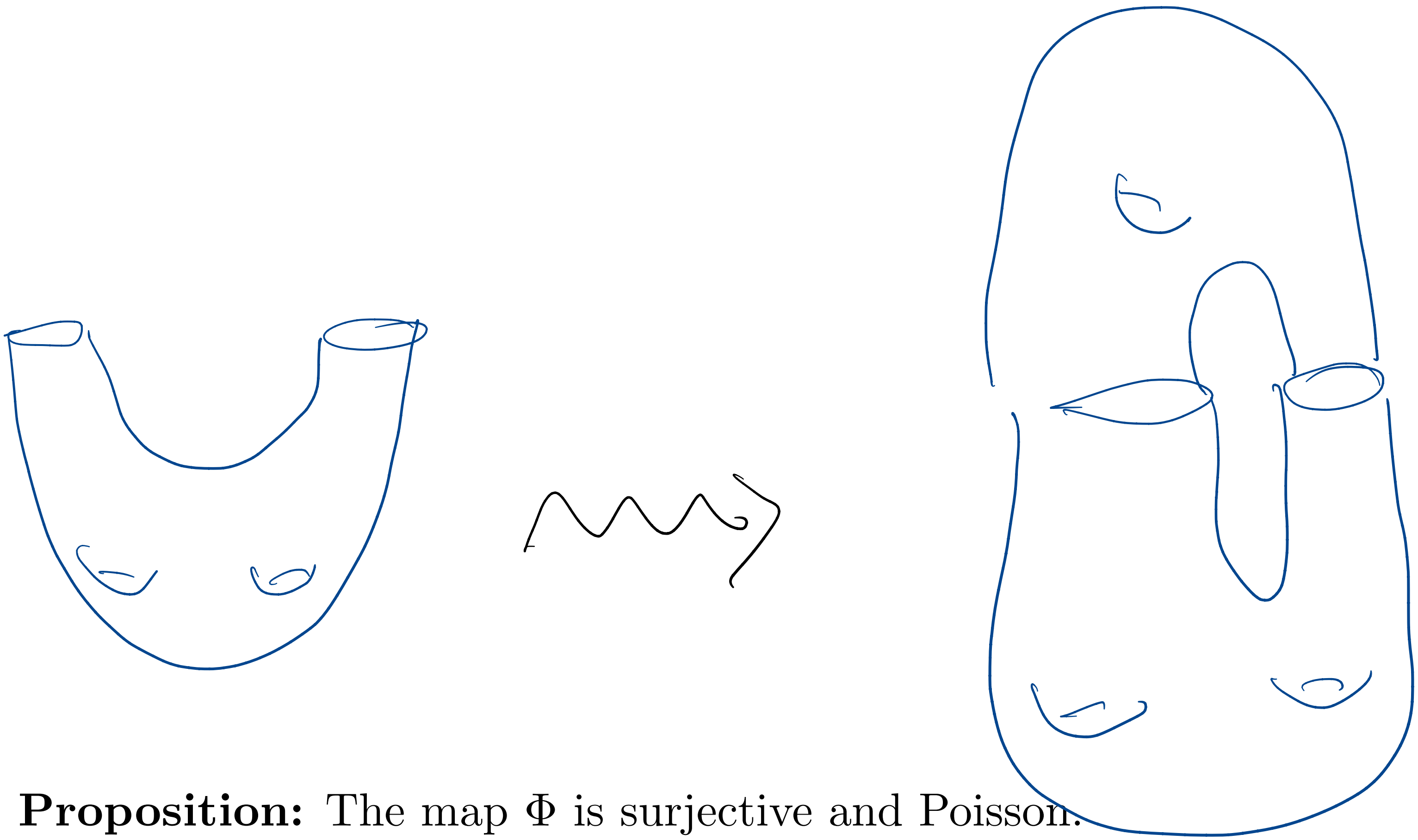
The map Φ is not surjective.

(d) Capping one boundary component with a genus 1 curve:



If G is semisimple, then the map Φ is surjective.

(e) Capping with an n-punctured genus 1 curve:



Proposition: The map Φ is surjective and Poisson.

7. Gluing via symplectic quotients.

Two collections of spaces whose quotients are character varieties:

(a) q-Hamiltonian spaces (Alekseev-Malkin-Meinrenken 1998):

• Let $D(G)$ be $G \times G$, with a $G \times G$-action. This space is called the double. Let $\mathbf{D}(G)$ also be $G \times G$, with a G-action. This space is called the internal fusion of the double.

• The action of $G \times G$ on the double $D(G)$ is by

$$(g_1, g_2) : (a, b) \mapsto (g_1 a g_2^{-1}, g_2 b g_1^{-1}).$$

• The 2-form on the double is

$$\frac{1}{2}(a^*\theta, b^*\bar{\theta}) + \frac{1}{2}(a^*\bar{\theta}, b^*\theta).$$

- The internal fusion $\mathbf{D}(G)$ is just $G \times G$ with the G action

$$g : (a, b) \mapsto (\mathrm{Ad}_g(a), \mathrm{Ad}_g(b)).$$

- The 2-form on the internal fusion $\mathbf{D}(G)$ is

$$\omega = \frac{1}{2}(a^*\theta, b^*\bar{\theta}) + \frac{1}{2}(a^*\bar{\theta}, b^*\theta) + \frac{1}{2}\Big((ab)^*\theta, (a^{-1}b^{-1})^*\bar{\theta}\Big).$$

- Here θ is the left Maurer-Cartan form and $\bar{\theta}$ is the right Maurer-Cartan form.

Here θ is often written as $\theta = a^{-1}da$ if $a \in G$.

• The q-Hamiltonian space $D(G)^r \times (\mathbf{D}(G))^g$ is $G^{2(g+r)}$ with the q-Hamiltonian action of $(z_0, \ldots, z_r) \in G^{r+1}$ given by

$$a_i \mapsto \mathrm{Ad}_{z_0} a_i$$

$$b_i \mapsto \mathrm{Ad}_{z_0} b_i$$

$$u_j \mapsto z_0 u_j (z_j)^{-1}$$

$$v_j \mapsto z_j v_j (z_j)^{-1}$$

$(j = 1, \ldots, r; \quad i = 1, \ldots, g)$

• This space is equipped with a 2-form (playing the role of symplectic form) and moment maps

$$\mu_j(a, b, u, v) = (v_j)^{-1} \qquad (j = 1, \ldots, r)$$

$$\mu_0(a, b, u, v) = \mathrm{Ad}_{u_1}(v_1) \ldots \mathrm{Ad}_{u_r}(v_r)[a_1, b_1] \ldots [a_g, b_g].$$

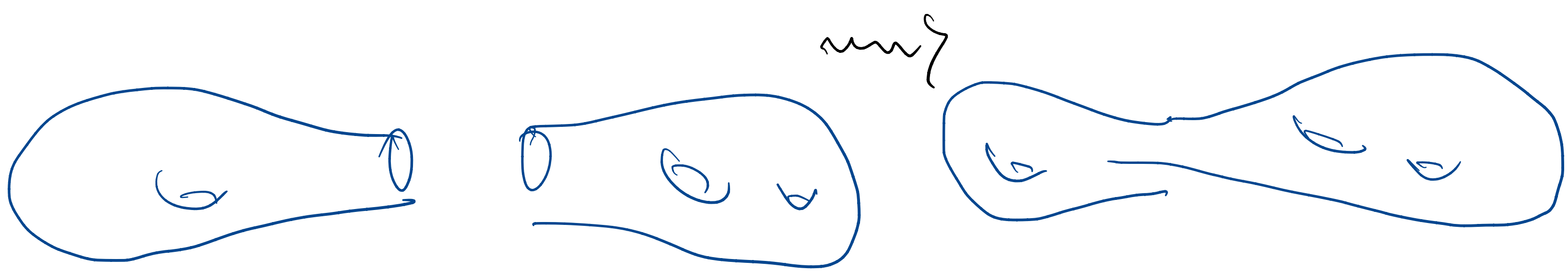
• Here we have used the notation a, b to refer to the tuple $(a_1, \ldots, a_g, b_1, \ldots, b_g)$.

• Similarly the notation u, v refers to the tuple $(u_1, \ldots, u_r, v_1, \ldots, v_r)$.

• $[a, b]$ denotes $aba^{-1}b^{-1}$ for $a, b \in H$.

• To form a Poisson manifold by gluing two boundary components of a connected surface, we set the values of the moment maps corresponding to those two components to be equal and then take the quotient by the diagonal action of the group.

• To form a Poisson manifold by gluing boundary components of two different surfaces, we set the moment maps on the two q-Hamiltonian spaces to be equal and take the quotient by the diagonal action of the group.



• This procedure can be iterated. If all boundary components are glued together, we recover a symplectic manifold.

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