# Duistermaat-Heckman measures for Hamiltonian groupoid actions

Luka Zwaan

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**Goal:** extend this to (suitable) Hamiltonian actions of symplectic groupoids.

# Outline

1. Basics of Hamiltonian actions of symplectic groupoids;
2. Duistermaat-Heckman measures for PMCTs;
3. Results:
    - The free case;
    - The locally free case.

# Symplectic groupoids and their Hamiltonian actions

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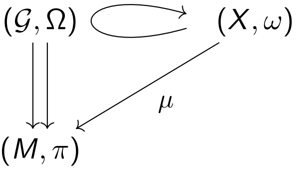


is called *Hamiltonian* if $a^*\omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\omega \in \Omega^2(\mathcal{G} \,{}_{\mathsf{s}}\!\times_{\mu} X)$.

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  - There is a polynomial Duistermaat-Heckman measure on the leaf space.

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  - Here $p : M \to B$ is projection to the leaf space, and vol associates to $b \in B$

$$\# \text{ connected comp's of isotropy} \times \text{symplectic volume of leaf};$$

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**Theorem (Crainic, Fernandes, Martínez-Torres)**

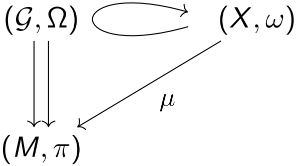
*We have*

$$\mu_{\mathrm{DH}}^{\Omega} = \mathrm{vol}^2 \cdot \mu_{\mathrm{aff}}.$$

*Moreover,* $\mathrm{vol}$ *is a polynomial on* $B$*.*

# Hamiltonian actions: the free case

- Suppose now we have a *free* Hamiltonian action

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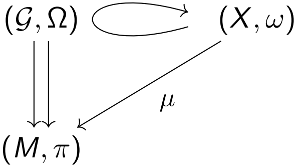


with $\mathcal{G} \rightrightarrows M$ regular, **s**-connected and **s**-proper and $\mu$ proper with connected fibres;

- Then $X/\mathcal{G}$ is a regular Poisson manifold of **s**-proper type and the integration $((X\ {}_{\mu}\!\times_{\mu} X)/\mathcal{G}, \omega \oplus -\omega)$ is symplectically Morita equivalent to $(\mathcal{G}, \Omega)$.

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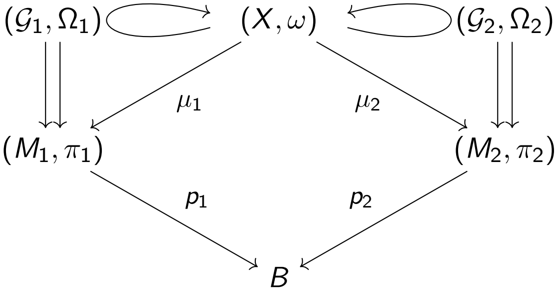
- One can show that given a symplectic Morita equivalence




between regular, **s**-connected and **s**-proper groupoids, we have

$$\mu_{\mathrm{DH}}^{\omega} := (p_i \circ \mu_i)_* \left( \frac{\omega^{\mathrm{top}}}{\mathrm{top}!} \right) = \mathrm{vol}_1 \cdot \mathrm{vol}_2 \cdot \mu_{\mathrm{aff}};$$

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- The leaf space of $\mathbb{L}$ can be thought of as the leaf space of $X/\mathcal{G}$.

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- We can once again define $\mu^{\omega}_{\mathsf{DH}}$ as the pushforward of $\dfrac{\omega^{\mathsf{top}}}{\mathsf{top}!}$.

# Hamiltonian actions: the locally free case

- The Morita equivalence

$$\begin{array}{ccc} \mu^*\mathcal{G} & \xrightarrow{\mathrm{pr}_2} & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ X & \xrightarrow{\mu} & M \end{array}$$

  induces an isomorphism of the leaf spaces *as integral affine orbifolds*;

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- Hence

$$\mu_{\mathrm{DH}}^{\omega} = (p \circ \mu)_* \left( \frac{\omega^{\mathrm{top}}}{\mathrm{top}!} \right).$$

# Hamiltonian actions: the locally free case

## Proposition

*We have*

$$\mu^{\omega}_{\mathrm{DH}} = \mathrm{vol} \cdot \mathrm{vol}_{\mathrm{red}} \cdot \mu_{\mathrm{aff}},$$

*where* $\mathrm{vol}_{\mathrm{red}}$ *associates to* $b \in B$

$\#$ *connected comp's of isotropy* $\times$ *symplectic volume of reduced space.*

*Moreover,* $\mathrm{vol}_{\mathrm{red}}$ *is a polynomial on* $B$*.*

Are there any questions?