Higher vector bundles

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based on joint work with G. Trentinaglia (IST)

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1. Representations as simplicial vector bundles

2. Filling horns in geometry and algebra

3. The semi-direct product construction

1.1 Lie groupoid representations

A **representation** $R : G \curvearrowright E$ of a Lie groupoid $G_1 \rightrightarrows G_0$ over a vector bundle $E \to G_0$ smoothly gives a linear isomorphism $R^g : E^x \to E^y$ for each $y \xleftarrow{g} x \in G_1$ and satisfies $R^{u_x} = \mathrm{id}_{E^x}$ and $R^h R^g = R^{hg}$

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Representations are scarce! Lack of adjoint rep. no Tannaka duality, ...

1.2 Representations up to homotopy [AriasAbad-Crainic]

Given G Lie groupoid, an m-simplex $g \in G_m$ on its nerve is

$$\underbrace{x_m \xleftarrow{g_m} x_{m-1} \xleftarrow{g_{m-1}} \dots \xleftarrow{g_{r+1}}}_{t_{m-r}(g)} x_r \underbrace{\xleftarrow{g_r} \dots \xleftarrow{g_2} x_1 \xleftarrow{g_1} x_0}_{s_r(g)}$$

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Let $E = \bigoplus_{n=0}^{N} E_n \to G_0$ be a graded vector bundle. A **RUTH** $R : G \curvearrowright E$ smoothly gives $R_m^g : E_n^{s_0(g)} \to E_{n+m}^{t_0(g)}$, $g \in G_m$, such that

- $R_1^{u(x)} = \mathrm{id}_{E^x}$, $x \in G_0$, and $R_m^{u_j(g)} = 0$, $g \in G_{m-1}$, $m > 1$
- $\sum_{i=1}^{m-1} (-1)^i R_{m-1}^{d_i(g)} = \sum_{r=0}^{m} (-1)^r R_{m-r}^{t_{m-r}(g)} R_r^{s_r(g)}$, $g \in G_m$

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Low degrees:
- R_0^x chain differential on the fiber E^x
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Paradigmatic examples: Adjoint and co-adjoint representations of a Lie groupoid (rule deformations, appear in Bott's spectral sequence)

1.3 Equivalent approaches (problem)

DGAs

A RUTH is the same as a degree 1 differential on $C^p(G, E) = \bigoplus_{m-n=p} \Gamma(G_m; t_0^* E_n)$ satisfying Leibniz and preserving normalized cochains [AriasAbad-Crainic]

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- N=1: A 2-term RUTH is the same as a VB-groupoid $q : V \to G$ endowed with a cleavage [GraciaSaz-Mehta]
- General: Today's Talk [dH-T]

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Classifying maps

- N=1: A 2-term RUTH is the same as a pseudo-functor $\rho : G \dashrightarrow GL(E_1 \oplus E_0)$ into the *general linear 2-groupoid* [dH-Stefani]
- General: yet to be done...

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Given G a Lie groupoid, a **higher vector bundle** $q : V \to G$ is a simplicial vector bundle over the nerve that is also a simplicial fibration.

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Proof:
- Builds on previous work of [Behrend, Getzler, Henriques, Zhu]
- uses new formulas for Dold-Kan;
- develops a theory of higher cleavages

1.5 Relation with literature

How it fits into the literature

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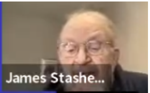
- (Possible) Solution to Block-Smith-Stasheff problem



Working in categories more appropriate for algebraic geometry,
ck and Smith define a *generalized Riemann–Hilbert*
respondence which is an equivalence of categories

$$Flat(M) \to hReps(Sing(M)),$$

en by the generalized holonomy of a flat $\mathbb{Z}$-graded connection.
re *hReps* denotes representations up to homotopy, although Block
d Smith refer to them as *infinity-local systems* on M.

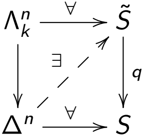
Block and Smith remark:
It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction.

2. Filling horns in geometry and algebra

- 2.1 Simplicial fibrations
- 2.2 Introducing higher cleavages
- 2.3 Coherent cleavages
- 2.4 The Dold-Kan correspondence
- 2.5 New formulas for the inverse

2.1 Simplicial Fibrations

A simplicial map between simplicial sets $q : \tilde{S} \to S$ is a **fibration** if the **relative horn map** $d^q_{n,k} : \tilde{S}_n \to \tilde{S}_{n,k} \times_{S_{n,k}} S_n$ is surjective



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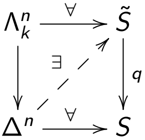

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Example

a) A locally trivial submersion $q : \tilde{M} \to M$ induces a fibration between their singular smooth simplices $S^\infty q : S^\infty \tilde{M} \to S^\infty M$ (N-strict $=$ 1-strict $=$ discrete fiber)

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$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\forall} & \tilde{S} \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow q \\ \Delta^n & \xrightarrow{\forall} & S \end{array}$$

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b) A groupoid morphism $q : \tilde{G} \to G$ is a *Grothendieck fibration* iff $Nq : N\tilde{G} \to NG$ is a fibration between nerves (1-strict)

2.2 Introducing higher cleavages

$q : \tilde{S} \to S$ a simplicial fibration. An n-**cleavage** $C_n \subset \tilde{S}_n$ is subset such that the relative horn maps $d^q_{n,k} : C_n \to \tilde{S}_{n,k} \times_{S_{n,k}} S_n$ are bijections for all $k < n$. It is **normal** if C contains degenerate simplices. A **cleavage** is a collection $C = \{C_n : n \geq 1\}$.

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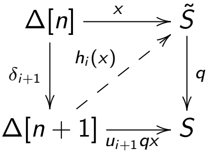
A Grothendieck fibration $q : \tilde{G} \to G$ splits by a cleavage C into a **fiber pseudo-functor** $F_C : G \dashrightarrow \mathrm{Gpds}$:

$$F(x) = q^{-1}(\mathrm{id}_x) \qquad F(g) : F(x) \to F(y) \text{ parallel transport}$$

Higher cleavages allow us to develop a higher version

2.3 Push-forward operators

$q : \tilde{S} \to S$ simplicial fibration, C cleavage. Given $x \in \tilde{S}_n$ and $i < n$, its **push-forward** $p_i(x) = d_i h_i(x)$ results of pushing-forward the i-th vertex of x to the left:



$$h_i \in C_{n+1}$$
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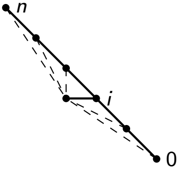

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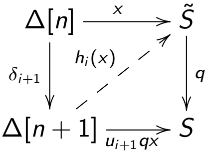

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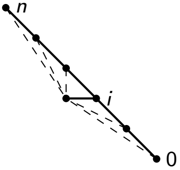
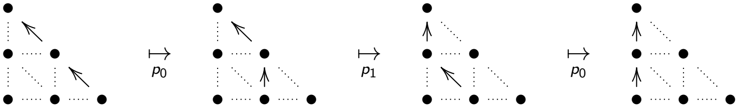


By iterated applications we can push-forward a simplex $x \in \tilde{S}_n$ to a new one $r(x) \in S_n$ over $t_0 q(x)$



2.4 The Dold-Kan correspondence

The **normalization** of a simplicial abelian group is a chain complex:

$$N : sAb \to Ch_{\geq 0}(Ab) \qquad NX_n = \bigcap_{i>0} \ker(d_i : X_n \to X_{n-1}) \quad \partial = d_0$$

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$$DK : Ch_{\geq 0}(Ab) \to sAb \qquad DK(Y)_n = \hom(NF\Delta[n], Y)$$

This is an instance of so-called *Kan extensions*.

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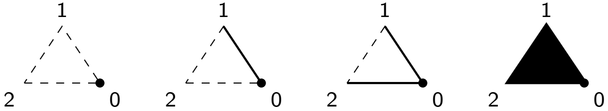
Explicit formula for DK popularized in the literature
(geometric meaning?)

2.5 New formulas for the inverse

Any simplex can be built by succesive horn-fillings from the 0-vertex.



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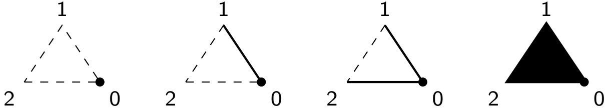

This inspires the following:

Proposition [dH-T]

The inverse of normalization can be described as follows:

$$DK(Y)_n = \bigoplus_{\substack{[k]\xrightarrow{\alpha}[n] \\ \alpha(0)=0}} Y_k \qquad \pi_\beta u_j = \pi_{\upsilon_j \beta} \quad \pi_\beta d_i = \pi_{\delta_i \beta}\ (i \neq 0)$$

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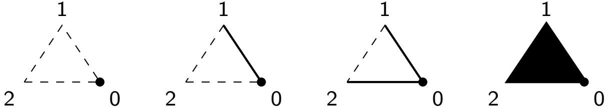

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Advantages of our formula: geometric meaning, allow generalization

3. The semi-direct product construction

- 3.1 The simplex vector bundles
- 3.2 Faces and degeneracies
- 2.3 Splitting a higher vector bundle
- 2.4 Splitting via coherent cleavages
- 2.5 Moving forward

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Given $R : G \curvearrowright E$ a RUTH, the *n*-**th simplex vector bundle** is

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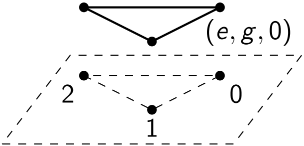

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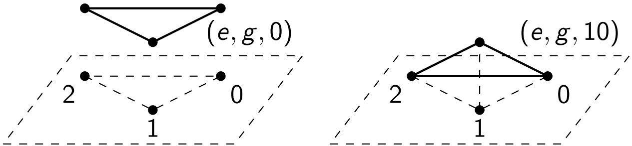

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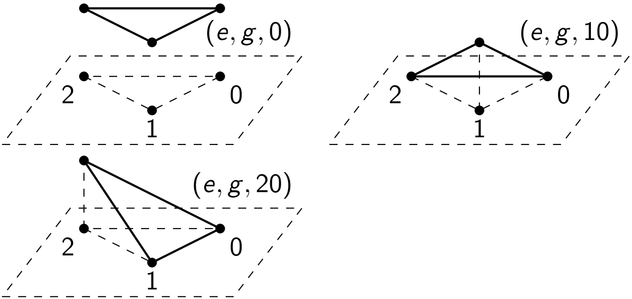

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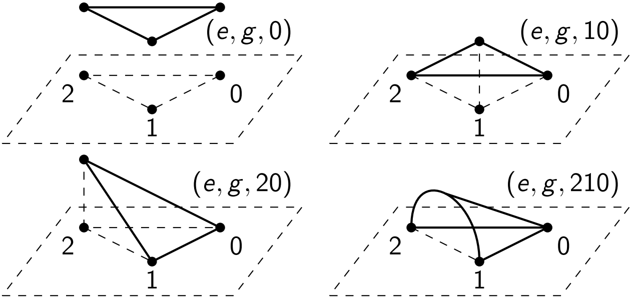

3.2 Faces and degeneracies

Positive faces and deg. are combinatorial: $\pi_\beta u_j = \pi_{\upsilon_j \beta}$, $\pi_\beta d_i = \pi_{\delta_i \beta}$
d_0 encodes R: $\pi_\beta \circ d_0 = \sum_{m+k=l+1} \pm R_m^{g\beta' \tau_m} \pi_{\beta' \sigma_k} - \sum_{0<i<l+1} (-1)^i \pi_{\beta' \delta_i}$

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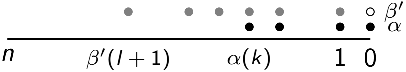
$$\pi_\beta d_0(e, \alpha, g) = \pm R_{l+1-k}^{g\beta' \tau_{l+1-k}}(e)$$



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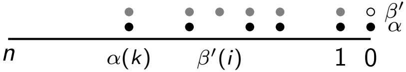

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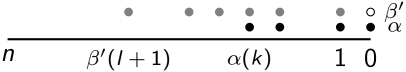

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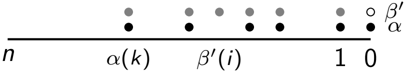

Theorem (First half)

$(G \ltimes_R E, d_i, u_j)$ is a higher vector bundle over G with core E.
The sub-bundles $C_n = \{v : v_{\iota_n} = 0\}$ form a normal coherent cleavage.

When $G = M$ this recovers new DK formula.
When $N = 1$ this recovers Grothendieck-GraciaSaz-Mehta construction.

3.3 Splitting a higher vector bundle

Given $q : V \to G$ a higher vector bundle, writing $K_n = \ker d^q_{n,0} : V_n \to d^*_{n,0} V_{n,0}$, and decomposing each simplex as succesive horn fillings from 0-vertex, we get

$$\phi_n : V_n \cong \bigoplus_{\substack{[k] \xrightarrow{\alpha} [n] \\ \alpha(0)=0}} a^* K_k$$

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This, combined with the push-forward operators induced by a normal cleavage C yields:

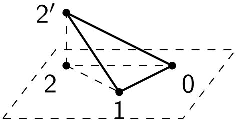
Direct sum decomposition

$q : V \to G$ higher vector bundle, C normal cleavage. There is an isomorphism $\phi_n : V_n \cong \bigoplus_{\substack{[k] \xrightarrow{\alpha} [n] \\ \alpha(0)=0}} x^*_{\alpha(k)} E_k$ where $E_k = K_k|_{G_0}$ **core**.

3.4 Splitting via coherent cleavages

The direct sum decomposition given by normal cleavage preserves the positive faces and the degeneracies (combinatorial)...

... but it may fail to preserve d_0!



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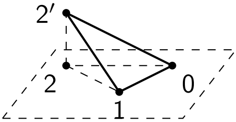

A cleavage C is **coherent** if the following holds true:

$$w \in C_{n+1} \text{ st } \begin{cases} d_i(w) \in C_n & 0 < i \\ s_k(w) \in C_k & 0 < k < n \\ s_0(w) = 0 \end{cases} \quad \Rightarrow \quad d_0(w) \in C_n$$

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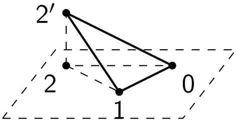

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Theorem (Second half)

$q : V \to G$ higher vector bundle, C normal coherent cleavage, then V is a semi-direct product of a representation up to homotopy.

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Thanks!!