Local structure of Lagrangians and shifted Poisson geometry

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Introduction

Suppose X is a smooth (C^∞) symplectic manifold and $L \subset X$ a smooth Lagrangian submanifold.

Theorem (Lagrangian neighborhood theorem)

There is a neighborhood of $L \subset X$ which is symplectomorphic to a neighborhood of the zero section $L \subset \mathrm{T}^* L$.

Totally false when working with holomorphic or algebraic symplectic structures. Does not work with formal or even first-order neighborhoods!

Example

Consider an elliptic fibration $f\colon S \to \mathbf{P}^1$ of a K3 surface. The surface S has a holomorphic symplectic structure and the fibers of f are Lagrangian. The generic fiber of f has no first-order splitting.

Holomorphic Lagrangian splittings

Let $L \subset X$ be a Lagrangian submanifold. Then we have an exact sequence

$$0 \longrightarrow \mathrm{T}_L \longrightarrow \mathrm{T}_X|_L \longrightarrow \mathrm{N}_L \cong \mathrm{T}^*_L \longrightarrow 0$$

which defines an extension class $\alpha \in \mathrm{H}^1(L, \mathrm{Sym}^2\,\mathrm{T}_L)$.

Where to go from here:

- If X is hyperKähler, from α one can extract a cubic form on $\mathrm{H}^0(L, \mathrm{T}^*_L)$. This defines a special Kähler structure on the moduli space of holomorphic Lagrangians in X (Hitchin).
- α is a bivector underlying a (-1)-*shifted Poisson structure*.

Shifted symplectic structures

In the context of derived algebraic geometry Pantev–Toën–Vaquié–Vezzosi introduced ***n*-shifted symplectic structures**: elements

$$\omega_2 \in \mathrm{H}^n(X, \wedge^2 \mathrm{T}^*_X)$$

satisfying

- Nondegeneracy condition: $\omega_2 \colon \mathrm{T}_X \to \mathrm{T}^*_X[n]$ is an isomorphism.
- Closure: $\mathrm{d}\omega_2 = 0$ holds up to coherent homotopy. Explicitly: there are forms $\omega_3, \omega_4, \dots$ such that

$$\mathrm{d}\omega_2 \overset{\omega_3}{\sim} 0, \qquad \mathrm{d}\omega_3 \overset{\omega_4}{\sim} 0, \qquad \dots$$

Calaque–Pantev–Toën–Vaquié–Vezzosi introduced ***n*-shifted Poisson structures**: elements $\pi_2 \in \mathrm{H}^{-n}(X, \wedge^2 \mathrm{T}_X)$ (for n even) or $\pi_2 \in \mathrm{H}^{-n}(X, \mathrm{Sym}^2 \mathrm{T}_X)$ (for n odd) such that

$$[\pi_2, \pi_2] = 0$$

holds up to coherent homotopy. Explicitly: there are polyvectors $\pi_3, \pi_4, \dots$ such that

$$[\pi_2, \pi_2] \overset{\pi_3}{\sim} 0, \qquad [\pi_2, \pi_3] \overset{\pi_4}{\sim} 0, \qquad \dots$$

The homotopies are interesting. E.g. if $\pi_2 = 0$, then $\pi_3 \in \mathrm{H}^{-2n-1}(X, \wedge^3 \mathrm{T}_X)$.

Shifted Lagrangian structures

Theorem (CPTVV, Pridham)

The inverse of an n-shifted symplectic structure ω defines an n-shifted Poisson structure π.

Given a map $f\colon L \to X$, where X has an n-shifted symplectic structure, one can define a **Lagrangian structure**: nullhomotopy $f^*\omega = 0$ such that the induced map $\mathrm{N}_L \to \mathrm{T}^*_L[n]$ is an isomorphism.

Theorem (Melani–S)

Suppose $f\colon L \to X$ has an n-shifted Lagrangian structure. Then there is a natural $(n-1)$-shifted Poisson structure on L.

Idea: just like one can invert n-shifted symplectic structures to get n-shifted Poisson structures, one can invert n-shifted Lagrangian structures to get n-shifted coisotropic structures. In particular, they consist of an n-shifted Poisson structure on X and an $(n-1)$-shifted Poisson structure on L.

Example (revisited). Let $L \subset X$ be an ordinary Lagrangian submanifold, so that $n = 0$. Then L carries a (-1)-shifted Poisson structure.

Symplectic groupoids

Recall: $\mathcal{G} \rightrightarrows X$ is a symplectic groupoid if:

- There is a multiplicative symplectic structure ω on $\mathcal{G}$.
- The unit section $X \to \mathcal{G}$ is Lagrangian. In particular, the Lie algebroid of $\mathcal{G}$ is T^*_X.

The anchor map

$$\rho\colon \mathrm{T}^*_X \longrightarrow \mathrm{T}_X$$

endows X with a Poisson structure π, so that $\mathcal{G}$ is an *integration* of the Poisson structure.

Theorem (S)

There is a 1-shifted Lagrangian structure on the quotient map $X \to [X/\mathcal{G}]$ determined by the symplectic structure ω on $\mathcal{G}$. The underlying 0-shifted Poisson structure on X is π.

L_∞ structure on $\mathrm{T}_X[-1]$

Let X be a complex manifold.

- Atiyah: the obstruction class for holomorphic (torsion-free) connections on T_X is

$$\mathrm{at}_X \in \mathrm{H}^1(X, \mathrm{T}_X \otimes \mathsf{Sym}^2(\mathrm{T}^*_X)).$$

- Kapranov: this is a Lie bracket underlying an L_∞ structure on $\mathrm{T}_X[-1]$ in $\mathrm{DCoh}(X)$. Namely, $\wedge^2(\mathrm{T}_X[-1]) \to \mathrm{T}_X[-1]$ is the same as map $\mathsf{Sym}^2(\mathrm{T}_X) \to \mathrm{T}_X[1]$.

Suppose further that X is holomorphic symplectic. Then:

- $\mathrm{at}_X \in \mathrm{H}^1(X, \mathsf{Sym}^3(\mathrm{T}_X))$.
- There is an IHX relation $\{\mathrm{at}_X, \mathrm{at}_X\} = 0 \in \mathrm{H}^2(X, \mathsf{Sym}^4(\mathrm{T}_X))$.

Looks like a part of a (-1)-shifted Poisson structure!

L_∞ structure on $\mathrm{T}_X[-1]$

If $\mathfrak{g}$ is an L_∞ algebra equipped with a nondegenerate pairing (cyclic structure), then $\mathrm{C}^\bullet(\mathfrak{g}) = \widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1])$ has a Poisson structure and the CE differential d is given by $\mathrm{d} = \{\Phi, -\}$ for a **potential**

$$\Phi \in \widehat{\mathrm{Sym}}^{\geq 3}(\mathfrak{g}^*[-1]).$$

Proposition

If X is symplectic, the Lie algebra $\mathrm{T}_X[-1]$ is cyclic. The potential for this cyclic L_∞ structure is the (-1)-shifted Poisson structure underlying the diagonal Lagrangian $\Delta\colon X \to X \times \overline{X}$.

The cubic term in Φ determines the bracket, so $\mathrm{at}_X \in \mathrm{H}^1(X, \mathrm{Sym}^3(\mathrm{T}_X))$ is the trivector of the (-1)-shifted Poisson structure.

Linearization of Poisson-Lie groups

Let G be a Poisson-Lie group. Often it comes as a part of a Manin triple:

- D is a Lie group with a nondegenerate pairing on $\mathrm{Lie}(D)$.
- $G \subset D$ is a Lagrangian subgroup.
- $G^* \subset D$ is a transverse Lagrangian subgroup, the Poisson-Lie dual.

To first order near $e \in G^*$ the Poisson structure looks like the Kirillov–Kostant–Souriau Poisson structure near $0 \in \mathfrak{g}^*$.

Question: is there a Poisson isomorphism of (formal) neighborhoods?

There is a natural 1-shifted symplectic structure on the quotient stack $[G\backslash D/G]$ and a Lagrangian structure on the inclusion of the unit $[\mathrm{pt}/G] \to [G\backslash D/G]$.

Proposition (S)

The Poisson-Lie group G^* is formally linearizable if, and only if, the 0-shifted Poisson structure on $[\mathrm{pt}/G]$ coming from the 1-shifted Lagrangian $[\mathrm{pt}/G] \to [G\backslash D/G]$ is zero.

Morse–Bott matrix factorizations

Let X be a smooth algebraic variety and $W\colon X \to \mathbf{C}$ an algebraic function. $\mathrm{MF}(X, W)$ is the 2-periodic dg category of *matrix factorizations*: diagrams

$$V_1 \underset{d}{\overset{d}{\rightleftarrows}} V_0$$

of coherent sheaves such that $\mathrm{d}^2 = W \cdot \mathrm{id}$.

Suppose the critical locus $Y \subset X$ of W is smooth. Then N_Y has a quadratic form given by the Hessian of W. It has the Stiefel–Whitney classes

$$w_1 \in \mathrm{H}^1(Y; \mathbf{Z}/2), w_2 \in \mathrm{H}^2(Y; \mathbf{Z}/2)$$

and we can consider the twisted derived category $\mathrm{DCoh}^{w_1, w_2}(Y)_{\mathbf{Z}/2}$ of 2-periodic coherent complexes.

The following was conjectured by Kapustin–Rozansky–Saulina (modulo the correction).

Theorem (Teleman)

There is a Maurer–Cartan element $\pi \in \Omega^{0,\bullet}(Y, \wedge^\bullet \mathrm{T}_Y)$, so that $\mathrm{MF}(X, W)$ is equivalent to the deformation of $\mathrm{DCoh}^{w_1, w_2}(Y)_{\mathbf{Z}/2}$ along π.

Morse–Bott matrix factorizations

Given the function $W\colon X \to \mathbf{C}$ we can consider the following objects:

- Its critical locus $Y = \mathrm{Crit}(W) \subset X$. This is a (possibly singular) subscheme of X.
- The derived critical locus $\mathrm{dCrit}(W)$. It is a derived scheme which has a (-1)-shifted symplectic structure.
- There is a natural map $\mathrm{Crit}(W) \to \mathrm{dCrit}(W)$.

Proposition

Suppose $\mathrm{Crit}(W)$ *is smooth. Then* $\mathrm{Crit}(W) \to \mathrm{dCrit}(W)$ *has a unique Lagrangian structure with* π *the underlying* (-2)*-shifted Poisson structure on* $\mathrm{Crit}(W)$*.*

This fits into the following program:

- For any (-1)-shifted symplectic scheme Z equipped with a spin structure there is a 2-periodic dg category MF_Z.
- $\mathrm{MF}_{\mathrm{dCrit}(W)} = \mathrm{MF}(X, W)$ and so $\mathrm{MF}_{\mathrm{T}^*[-1]Y} = \mathrm{DCoh}(Y)_{\mathbf{Z}/2}$.

The above proposition describes $\mathrm{dCrit}(W)$ as a twist of $\mathrm{T}^*[-1]Y$ along π.