Math 121 Final Exam May 15, 2013

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need not simplify algebraically complicated answers. However, numerical answers such as \( \sin \left( \frac{\pi}{6} \right) \), \( 4^{\ln 2} \), \( e^{\ln 5} \), \( e^{3 \ln 3} \), \( \arctan(\sqrt{3}) \), or \( \cosh(\ln 3) \) should be simplified.

• Please show all of your work and justify all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, \(+\infty\) or \(-\infty\), or Does Not Exist.

   \[
   \begin{align*}
   (a) & \quad \lim_{x \to 0} \frac{3xe^x - \arctan(3x)}{x + \ln(1 - x)} \\
   (b) & \quad \lim_{x \to \infty} \left( \cosh \left( \frac{1}{x} \right) - \frac{5}{x} \right)^x
   \end{align*}
   \]

2. [30 Points] Evaluate each of the following integrals.

   \[
   \begin{align*}
   (a) & \quad \int \frac{e^x}{(e^{2x} + 4)^{\frac{3}{2}}} \, dx \quad \text{Hint: } e^{2x} = (e^x)^2 \\
   (b) & \quad \int x \arctan x \, dx \\
   (c) & \quad \int x \arcsin x \, dx \\
   (d) & \quad \int \frac{x^4 + 5x^2 - x + 3}{x^3 + 3x} \, dx
   \end{align*}
   \]

3. [20 Points] For each of the following improper integrals, determine whether it converges or diverges. If it converges, find its value.

   \[
   \begin{align*}
   (a) & \quad \int_{6}^{\infty} \frac{1}{x^2 - 10x + 28} \, dx \\
   (b) & \quad \int_{0}^{9} \frac{1}{(x - 1)^{\frac{3}{2}}} \, dx
   \end{align*}
   \]

4. [15 Points] Find the sum of each of the following series (which do converge):

   \[
   \begin{align*}
   (a) & \quad \sum_{n=1}^{\infty} \frac{(-1)^n \, 3^{2n-1}}{4^{2n+1}} \\
   (b) & \quad \sum_{n=0}^{\infty} \frac{(-1)^n \, 2^n}{n!} = 1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - \ldots \\
   (c) & \quad \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n)!}
   \end{align*}
   \]

5. [30 Points] In each case determine whether the given series is absolutely convergent, conditionally convergent, or diverges. Justify your answers.

   \[
   \begin{align*}
   (a) & \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n + 7} \\
   (b) & \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \\
   (c) & \quad \sum_{n=1}^{\infty} \frac{n + 3}{\ln(n + 3)}
   \end{align*}
   \]

   \[
   \begin{align*}
   (d) & \quad \sum_{n=1}^{\infty} \frac{(-1)^n \, 3 + n^2}{n^7 + 4} \\
   (e) & \quad \sum_{n=1}^{\infty} \frac{\arctan(6n)}{6^n} + \frac{6}{n^6} \\
   (f) & \quad \sum_{n=1}^{\infty} \frac{(-1)^n \, n^n (256^n \, (n!)^3)}{\pi^n (4n)!}
   \end{align*}
   \]
6. [15 Points] Find the Interval and Radius of Convergence for the following power series \( \sum_{n=0}^{\infty} \frac{(-1)^n (5x + 1)^n}{(n^2 + 1)^9} \). Analyze carefully and with full justification.

7. [10 Points] (a) Write the MacLaurin Series for \( f(x) = x^5 \sin(x^3) \).
(b) Use this series to determine the eighth and ninth derivatives of \( f(x) = x^5 \sin(x^3) \) at \( x = 0 \).
(Hint: Do not compute out those derivatives manually.)
(Hint: Write out the definition of the MacLaurin Series for any \( f(x) \).)

8. [15 Points] Please analyze with detail and justify carefully.
(a) Write the MacLaurin series representation for \( f(x) = xe^{-x^7} \). Your answer should be in sigma notation \( \sum_{n=0}^{\infty} \).
(b) Use the MacLaurin series representation for \( f(x) = xe^{-x^7} \) from Part(a) to Estimate \( \int_{0}^{1} xe^{-x^7} \, dx \) with error less than \( \frac{1}{10} \).
Justify in words that your error is indeed less than \( \frac{1}{10} \).

9. [15 Points] (a) Consider the region bounded by \( y = \arcsin x, \ y = \frac{\pi}{2}, \ x = 0 \) and \( x = 1 \). Rotate the region about the line \( x = 5 \). Set-Up but DO NOT EVALUATE the integral representing the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.
(b) Consider the region bounded by \( y = e^x, \ y = \ln x, \ x = 1 \) and \( x = 2 \). Rotate the region about the \( y \)-axis. COMPUTE the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [20 Points] Parametric Curves
(a) Consider the Parametric Curve represented by \( x = \frac{t^3}{3} - \frac{e^{2t}}{2} \) and \( y = 2te^t - 2e^t \). COMPUTE the arclength of this parametric curve for \( 0 \leq t \leq 1 \).
(b) Consider the Parametric Curve represented by \( x = \cos^3 t \) and \( y = \sin^3 t \). COMPUTE the surface area obtained by rotating this curve about the \( y \)-axis, for \( 0 \leq t \leq \frac{\pi}{2} \).

11. [15 Points] Compute the area bounded outside the polar curve \( r = 2 + 2 \cos \theta \) and inside the polar curve \( r = 6 \cos \theta \). Sketch the Polar curves.