Amherst College
Department of Mathematics

MATH 260 | IN-CLASS FINAL EXAM | SPRING 2014

NAME:____

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable
- Answers must be clearly labeled in the spaces provided after each question.
- Please cross out or fully erase any work that you do not want graded.
- The point value of each question is indicated after its statement.
- No books or other references are permitted.
- Calculators are not allowed and you must show all your work.

Grading - For Administrative Use Only

Question:	1	2	3	4	5	6	7	8	Total
Points:	16	14	10	12	12	14	10	12	100
Score:									

1. Suppose a second-order, linear, constant-coefficient ODE of the form [16]

$$y'' + a\,y' + b\,y = f(t) \tag{1}$$

has the general solution

$$y(t) = e^{-4t}\left(C_1\cos(3t) + C_2\sin(3t)\right) + 2\,\sin(4t).$$

   (a) What are the values of a and b in (1)?

   (b) What is the forcing function, $f(t)$?

   (c) If equation (1) represents a mass-spring system with a mass of 10 kg with displacement measured in meters and time measured in seconds, what would be the:

      (i) spring constant?

      (ii) damping constant?

   (d) What is the steady-state (a.k.a. long-term) solution?

2. Let $H(t)$ denote the temperature (in Kelvin) of a star at time t. The rate of change of temperature with respect to time is given by the heat generated due to nuclear fusion minus the heat lost due to radiation. [14]

   Assume that the heat generated is proportional to the square of the temperature and the heat radiated is proportional to the temperature raised to the fourth power. Denote the positive constants of proportionality for fusion and radiation with n and r respectively.

   (a) Write a differential equation for $H(t)$.

   (b) Consider the differential equation you found in part (a) with $n = 4$ and $r = 1$. Find all equilibrium temperatures for the star, and classify each as a source, sink, or node.

   (c) In part (b), we assumed $n = 4$ and $r = 1$, which are not particularly realistic values. However, the qualitative behavior of the DE is the same for any positive values of n and r. For arbitrary positive values of n and r and $H(0) > 0$, describe the behavior of $H(t)$ as $t \to \infty$. (Your answer should involve n and r.)

3. Put a check mark in the box if that term accurately describes the differential equation. YOU DO NOT HAVE TO SOLVE THE DE. [10]

	second-order?	linear?	separable?	homogeneous? (a.k.a. unforced or undriven)?
(a) $xy' = y$				
(b) $y'x = 3x^2 + 2$				
(c) $y'' = y^2$				
(d) $6y'' + 3xy' + y = 0$				
(e) $yy' = 4x - 1$				

4. Consider the initial value problem. [12]

$$\frac{dy}{dx} = xy + 3x, \qquad y(0) = 2.$$

(a) Solve the IVP using the method of separation of variables.

(b) Solve the IVP using an integrating factor. (Do not leave any integrals or unknown constants in your final solution.)

5. Consider the one-parameter family: [12]

$$\frac{dx}{dt} = 2 + \alpha x + 2x^2$$

(a) Locate the bifurcation value(s).

(b) Draw the phase lines for values of α slightly smaller than, slightly larger than, and at the bifurcation value(s).

6. Consider the following system of differential equations: [14]

$$\frac{dx}{dt} = -4x + 4y$$
$$\frac{dy}{dt} = 2x + 3y$$

(a) Find the general solution.

(b) Sketch the phase plane.

(c) Classify the origin as a sink, source, saddle, or center.

(d) Is the origin a stable or unstable equilibrium point?

7. The general existence and uniqueness theorem can be applied to any first order IVP satisfying the proper assumptions. Consider the linear IVP [10]

$$y' + \left(\frac{x+2}{x-1}\right) y = 0; \qquad y(0) = 3$$

but DO NOT SOLVE IT.

(a) Using the general existence and uniqueness theorem on the IVP above, what can you conclude about the solution?

(b) Now use the linear version of the existence and uniqueness theorem on the same IVP above. What more can you conclude about the solution?

8. Consider the following system of differential equations: [12]

$$\frac{dx}{dt} = -2x - y$$
$$\frac{dy}{dt} = 2x - 5y$$

(a) Show that the following are both solutions to this system:

$$Y_1(t) = \begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix} \quad \text{and} \quad Y_2(t) = \begin{bmatrix} e^{-4t} \\ 2e^{-4t} \end{bmatrix}$$

(b) Show that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions.

(c) Find the solution to the corresponding initial value problem given the initial condition:

$$Y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$