Math 211: Multivariable Calculus, Spring 2015
Final Exam
Wednesday May 13, 2015

- You have 3 hours for this exam.
- You may not use books, notes, calculators, cell phones or any other aids.
- You must explain your answers clearly and completely to get full credit.
- Follow directions carefully: if the question tells you to use a particular method or definition, then you must use it in order to get credit.
- There are 12 questions and 18 pages (including this one).
- The total number of points available is 100.

1. (a) (5 points) Calculate the equation of the tangent plane to the surface

$$x^2y + y^2z + z^2x = 7$$

at the point $(1, 1, 2)$.

(b) (2 points) Find parametric equations for the line passing through the point $(1, 1, 2)$ that is perpendicular to the plane you found in part (a).

2. At a particular time, the temperature (in degrees Fahrenheit) at location (x, y) (measured in miles) is given by the formula

$$T(x, y) = (x - 1)(y + 2) + 80$$

(a) (3 points) Amherst is located at the point $(4, 3)$. Find the directional derivative of the temperature function at Amherst in the direction of the vector $\langle 3, -2\rangle$. (Include appropriate units for your answer.)

(b) (2 points) Starting at Amherst, in which direction should someone travel in order to increase the temperature as fast as possible? (Give your answer as a vector and explain how you got it.)

(c) (2 points) What is the minimum value of the directional derivative of the temperature function at Amherst?

(d) (2 points) Use a linear approximation to estimate the temperature at the point $(4.02, 2.97)$.

3. (10 points) This question concerns the function

$$f(x, y) = \begin{cases} \dfrac{2x^6}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

At which points in $\mathbb{R}^2$ is the function f differentiable? Explain your answers clearly.

4. (3 points) Let $\mathbf{r}(t)$ be the vector-valued function that represents the position of an object moving with constant speed. Prove that, for any value of t, the acceleration vector $\mathbf{r}''(t)$ is perpendicular to the velocity vector $\mathbf{r}'(t)$.

5. Consider the 'conical spiral' given by

$$\mathbf{r}(t) = \left\langle e^t \cos t, e^t \sin t, e^t \right\rangle.$$

   (a) (4 points) Calculate the arc length along this curve between the points $(1, 0, 1)$ and $(e^{2\pi}, 0, e^{2\pi})$.

   (b) (4 points) Calculate the vector curvature of this curve at the point $(1, 0, 1)$.

6. (a) (2 points) Explain how you know that the function

$$f(x, y) = x(y + 7)$$

   has an absolute maximum value and an absolute minimum value when restricted to the region

$$\{(x, y) \mid x^2 + y^2 \leq 9\}.$$

   (b) (8 points) Calculate the absolute maximum and absolute minimum values referred to in part (a).

7. (8 points) Let R be the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(2, 1)$. Evaluate the integral

$$\iint_R y \, dA.$$

8. (8 points) Use the change of variables

$$u = x + y, \quad v = 2x - y$$

   to find the area of the region in the xy-plane bounded by the curve

$$(x + y)^2 + (2x - y)^2 = 4.$$

9. (8 points) Calculate the volume of the solid region bounded by the paraboloid

$$z = x^2 + y^2$$

   and the cone

$$z = 2 - \sqrt{x^2 + y^2}.$$

10. (3 points each) Decide if each of the following vector fields $\mathbf{F}$ is conservative and, if so, find a function f such that $\mathbf{F} = \nabla f$. Give reasons for your answers.

   (a) $\mathbf{F}(x, y) = \langle y \cos x - 1, \sin x + 2y \rangle$

   (b) $\mathbf{F}(x, y) = \langle \ln(y), x \rangle$ on the region where $y > 0$

(c) $\mathbf{F}(x, y) = \left\langle \dfrac{y}{x^2 + y^2}, \dfrac{-x}{x^2 + y^2} \right\rangle$ on the region where $(x, y) \neq (0, 0)$.

11. (4 points each) Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ in each of the following cases:

(a) $\mathbf{F} = \nabla f$ where $f(x, y) = \ln(x^2 + y^2)$, C is the curve with parametrization

$$\mathbf{r}(t) = \left\langle e^{t^2 - t}, (t + 1)^2 \right\rangle, \quad 0 \leq t \leq 1.$$

(b) $\mathbf{F}(x, y) = \langle y^2, x \rangle$, C is the part of the curve $y = x^3$ between $(0, 0)$ and $(1, 1)$ oriented from $(0, 0)$ to $(1, 1)$.

(c) $\mathbf{F}(x, y) = \langle -\sqrt{y}, \sqrt{x} \rangle$, C is the boundary of the square $[0, 1] \times [0, 1]$ oriented clockwise.

12. (8 points) Use Green's Theorem to calculate the area of the ellipse given by the parametrization

$$\mathbf{r}(t) = \left\langle \sin t - \cos t, \frac{\sin t + \cos t}{2} \right\rangle, \quad 0 \leq t \leq 2\pi.$$