Math 385 Final Exam Spring 2015

Rules: You may use your notes and your textbook, but no other resources. You must work alone. Due Thurs., May 14, at noon.

1. (12 points) Find a deduction of Pab from $\{\forall x Pxa, \forall x(Pax \to x = b)\}$. (In this problem, a and b are constant symbols and P is a 2-place predicate symbol. As always, Proof Check output is OK, and you can get partial credit for a proof that a deduction exists.)

2. (8 points each) In this problem we use a language with two 1-place predicate symbols P and Q. Are the following statements true or false? Justify your answers.

   (a) $\forall x Px \to \exists y Qy \vDash \forall x(Px \to \exists y Qy)$.

   (b) $\forall x \exists y(Px \to Qy) \vDash \exists y \forall x(Px \to Qy)$.

3. (8 points each)

   (a) Show that for any formulas α and β, $(\forall x\alpha \vee \forall x\beta) \vDash \forall x(\alpha \vee \beta)$.

   (b) Give an example of a language, and formulas α and β in that language, such that $\forall x(\alpha \vee \beta) \nvDash (\forall x\alpha \vee \forall x\beta)$. Justify your answer.

   (c) Show that if x does not occur free in α then $\forall x(\alpha \vee \beta) \vDash (\forall x\alpha \vee \forall x\beta)$.

4. (10 points) Suppose Σ is a set of sentences, and for every structure $\mathfrak{A}$ there is some $\sigma \in \Sigma$ such that $\vDash_{\mathfrak{A}} \sigma$. Prove that there are finitely many sentences $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma$ such that $\vDash \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n$.

5. (8 points each) Suppose that A is a set of sentences in the language of number theory, $A_E \subseteq A$, A is consistent, and $\#A$ is representable. We will say that a sentence σ is *efficiently deducible* from A if there is a deduction of σ from A whose Gödel number is less than $100^{(100^{\#\sigma})}$. We will write $A \vdash_{\mathrm{e}} \sigma$ to indicate that σ is efficiently deducible from A, and $\mathrm{Cn_e}(A) = \{\sigma \mid \sigma \text{ is a sentence and } A \vdash_{\mathrm{e}} \sigma\}$.

   (a) Show that $\#\mathrm{Cn_e}(A)$ is representable.

   Thus, there is a formula $\varphi(v_1)$ such that for every natural number n, if $n \in \#\mathrm{Cn_e}(A)$ then $A_E \vdash \varphi(S^n 0)$, and if $n \notin \#\mathrm{Cn_e}(A)$ then $A_E \vdash \neg\varphi(S^n 0)$. By the fixed point lemma, there is a sentence σ such that

   $$A_E \vdash \sigma \leftrightarrow \neg\varphi(S^{\#\sigma}0).$$

   (b) Show that $A \nvdash_{\mathrm{e}} \sigma$, but $A \vdash \sigma$.

6. (12 points) In this problem we use a language with $=$ and a 1-place function symbol f. Let $\mathfrak{A}$ be the structure for this language defined as follows: $|\mathfrak{A}| = \mathbb{N}$, and for all $n \in \mathbb{N}$,

$$f^{\mathfrak{A}}(n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

(Note: 0 is even.) Show that $\mathrm{Th}(\mathfrak{A})$ is decidable. Hint: First find a decidable set $\Sigma \subseteq \mathrm{Th}(\mathfrak{A})$ that describes important features of $\mathfrak{A}$. Then show that $\mathrm{Cn}(\Sigma)$ is $\aleph_0$-categorical and apply the Łoś-Vaught test to show that $\mathrm{Cn}(\Sigma) = \mathrm{Th}(\mathfrak{A})$. Your proof that $\mathrm{Cn}(\Sigma)$ is $\aleph_0$-categorical can be fairly informal—you can just explain informally how to match up any two denumerable models of $\mathrm{Cn}(\Sigma)$.

7. (10 points) Suppose we expand first-order logic by adding a new quantifier symbol Q. In other words, we add the following clause to the definition of formula: If α is a formula and x is a variable, then $Qx\alpha$ is a formula. Also, we add the following clause to the definition of $\models_{\mathfrak{A}}$: $\models_{\mathfrak{A}} Qx\alpha[s]$ iff there are infinitely many $d \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \alpha[s(x|d)]$. In other words, "$Qx\alpha$" means "for infinitely many x, α." Consider a language that has this new quantifier (as well as all the other symbols of first-order logic), but no predicate symbols other than equality, and no function or constant symbols. Show that the Compactness Theorem is false for this language. In other words, show that there is a set of sentences Σ such that every finite subset of Σ is satisfiable, but Σ is not satisfiable.