Math 355 Introduction to Analysis, Spring 2016
Final Exam
May 11

1. (15 points) The sequence (a_n) is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ if $n \geq 1$.
   (a) (3 points) Prove that $a_n < a_{n+1}$ for all $n \geq 1$.
   (b) (2 points) Prove that $a_n < 2$ for all $n \geq 1$.
   (c) (10 points) Prove that $\lim_{n\to\infty} a_n$ exists and find its value. Justify all your claims.

2. (10 points)
   (a) (5 points) Prove that if the supremum of a nonempty set A exists, then it is unique.
   (b) (5 points) Does the closed interval $[0, 1]$ have the same cardinality as the open interval $(0, 1)$? Prove your claim.

3. (10 points)
   (a) (3 points) State the Bolzano-Weierstrass Theorem.
   (b) (7 points) Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

4. (10 points)
   (a) (4 points) State the Heine-Borel Theorem (the three equivalent statements for compact sets).
   (b) (6 points) Suppose the set K is compact and nonempty. Show that $\sup K$ exists and is an element of K.

5. (10 points)
   (a) (3 points) What does it mean to say f is uniformly continuous on $\mathbb{R}$?
   (b) (2 points) What does it mean to say the sequence (a_n) of real numbers is Cauchy?
   (c) (5 points) Suppose f is uniformly continuous on $\mathbb{R}$ and (x_n) is a Cauchy sequence of real numbers. Prove that $(f(x_n))$ is also Cauchy.

6. (10 points)

   (a) (3 points) State the generalized Mean Value Theorem.

   (b) (7 points) Prove that if $h : [0, a] \to \mathbb{R}$ is twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \le M$ for all $x \in [0, a]$, then $|h(x)| \le Mx^2/2$ for all $x \in [0, a]$.

7. (10 points)

   (a) (5 points) Prove that $f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \cap \mathbb{I} \end{cases}$ is not Riemann-integrable.

   (b) (5 points) Prove that if g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

8. (15 points)

   (a) (2 points) State the Weierstrass M-test.

   (b) (3 points) State the Integrable Limit Theorem.

   (c) (10 points) For each $n \in \mathbb{N}$, let $h_n(x) = \begin{cases} 1/3^n & 1/3^n < x \le 1 \\ 0 & 0 \le x \le 1/3^n \end{cases}$, and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H$.

9. (10 points)

   (a) (3 points) State the Term-by-Term Differentiability Theorem.

   (b) (2 points) State the Term-by-Term Continuity Theorem.

   (c) (5 points) Let

   $$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

   Prove that h is differentiable on $[-5, 5]$. Also prove that the derivative function h' is continuous on $[-5, 5]$.