Math 211-01 Multivariable Calculus, Spring 2017
Final Exam
May 11

1. (10 points)

   (a) (5 points) The point $(1,1)$ is an intersection of the graphs of $y = x$ and $y = x^4$. Express the acute angle between the tangent lines to these two graphs at $(1,1)$ as an inverse cosine.

   (b) (5 points) Find the distance between the parallel lines $x = y = z$ and $x - 1 = y - 2 = z - 3$.

2. (10 points)

   (a) (5 points) Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = 2xy$. Compute the curvature of of the curve C at $(1, \frac{1}{2}, \frac{1}{3})$.
   Hint: Write down a parametrization $\vec{r}(t)$ of C first. Start by letting $x = t$.

   (b) (5 points) Find the arc length of the curve $\vec{r}(t) = \langle \cos(4t), \sin(4t), 3t \rangle$, $0 \le t \le \pi$.

3. (10 points)

   (a) (5 points) We are given a partial derivative of a function $f(x,y)$:

   $$f_y(x,y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Compute $f_{yx}(0,0)$ and $f_{yy}(0,0)$.

   (b) (5 points) Find a unit vector $\vec{u}$ such that the directional derivative of $f(x,y) = x \ln y + x$ at the point $(1,1)$ in the direction of $\vec{u}$ is equal to 0.

4. (10 points)

   (a) (5 points) Is the function $f(x,y) = \begin{cases} \dfrac{xy^5}{x^2 + y^{10}} & \text{if } (x,y) \neq (0,0) \\ \frac{1}{2} & \text{if } (x,y) = (0,0) \end{cases}$ continuous at $(0,0)$? Prove your claim.

   (b) (5 points) Consider the vector field $\vec{F} = \langle x, y, 0 \rangle$.

      i. (2 points) Is $\vec{F}$ conservative? Explain.

      ii. (3 points) Is there another vector field $\vec{G}$ on $\mathbb{R}^3$ such that $\vec{F}$ is the curl of $\vec{G}$? Explain.

5. (10 points)

   (a) (7 points) Use the Lagrange multiplier method to find the absolute maximum and minimum values of the function $f(x, y) = xy - 1$ subject to the constraint $x^2 + y^2 = 2$. State all points where the extrema occur as well as the maximum and minimum values.

   (b) (3 points) Find the absolute maximum and minimum values of the function $f(x, y)$ in (a) over the closed disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 2\}$. The answer in (a) is useful here.

6. (10 points)

   (a) (6 points) Sketch the curve $r = \sqrt{2}\sin(2\theta)$ and compute the area of the region it bounds.

   (b) (4 points) Rewrite the following triple integral in the order $dydzdx$.

$$\int_0^1 \int_0^z \int_0^z f(x, y, z) dx dy dz.$$

7. (10 points)

   (a) (5 points) Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Find the value of I.
   Hint: Compute I^2 first.

   (b) (5 points) Use spherical coordinates to compute $\iiint_E z dV$ where E is the solid region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{3(x^2 + y^2)}$.

8. (10 points)

   (a) (5 points) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle e^{y^2 z}, 2xyze^{y^2 z}, xy^2 e^{y^2 z} \rangle$ and C is the polygonal line segment from $(0, 0, 0)$ to $(1, 0, 0)$ and then from $(1, 0, 0)$ to $(1, 1, 1)$. *Hint: Consider using the Fundamental Theorem of Line Integral.*

   (b) (5 points) Compute the surface integral $\iint_S x dS$ where S is the part of the cylinder $y^2 + z^2 = 4$ which is between the planes $x = 0$ and $x + y = 4$.

9. (10 points)

   (a) (5 points) Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y, z \rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ below $z = 4$ which is oriented downward.

   (b) (5 points) Use Stokes' Theorem to evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ where $\vec{F}(x, y, z) = \langle e^{xy}, e^{xz}, x^2 z \rangle$ and S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$, $y \geq 0$, oriented in the direction of the negative y-axis.

10. (10 points)

(a) (5 points) Use the Divergence Theorem to evaluate

$$\iint_S 2x^2 + y^2 + 5z dS,$$

where S is the sphere $x^2 + y^2 + z^2 = 9$.

Hint: Rewrite this surface integral as a surface integral of a vector field first.

(b) (5 points) The planar region D is defined by

$$\{(x, y) \in \mathbb{R}^2 | g_1(x) \leq y \leq g_2(x), a \leq x \leq b\},$$

where $y = g_1(x)$ and $y = g_2(x)$ are two smooth functions defined on $\mathbb{R}$. If $P(x, y)$ is a real-valued smooth function defined on $\mathbb{R}^2$, prove that

$$\int_C P dx = \iint_D -P_y dA,$$

where C is the counterclockwise-oriented boundary curve of D. **You cannot use Green's theorem, because you are actually proving part of it.**